Goals for the week

- understand the relationship between primal and dual programs.
- know how to convert an LP from primal to dual.
- know how to analyze the sensitivity of a BFS to changes in input data.
- understand complementary slackness.

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1 Duality

As we have seen in lecture, for every linear program there is a closely related alternative program called the ‘dual’. The relationship between these two programs is not only fascinating, but also very useful. As we shall see, it can be useful in gaining an intuition for the meaning of a program and to perform 'sensitivity analysis.' Moreover, as we will see in upcoming lectures it can also used to improve our Simplex algorithm.

1.1 Two parallel formulations

Given a program in 'standard equality form':

\[
\begin{align*}
\text{max} & \quad c^T x \\
\text{s.t.} & \quad Ax = b \\
& \quad x \geq 0
\end{align*}
\]

Our dual will have the form:

\[
\begin{align*}
\text{min} & \quad b^T y \\
\text{s.t.} & \quad A^T y \geq c \\
& \quad y \text{ free}
\end{align*}
\]

The constraints in the primal correspond to variables in the dual, and vice versa. In this case, we can draw the following parallels:

<table>
<thead>
<tr>
<th>Primal</th>
<th>Dual</th>
</tr>
</thead>
<tbody>
<tr>
<td>= Constraints</td>
<td>Free Variables</td>
</tr>
<tr>
<td>Non-negative Variables</td>
<td>( \geq ) constraints</td>
</tr>
</tbody>
</table>

In fact, the parallel formulations can be extended to LPs that are not in standard equality form. The following table specifies the equivalences between the primal and the dual in a general setting:

<table>
<thead>
<tr>
<th>Primal (maximize)</th>
<th>Dual (minimize)</th>
</tr>
</thead>
<tbody>
<tr>
<td>ith constraint ( \leq )</td>
<td>ith variable ( \geq 0 )</td>
</tr>
<tr>
<td>ith constraint ( = )</td>
<td>ith variable free</td>
</tr>
<tr>
<td>ith constraint ( \geq )</td>
<td>ith variable ( \leq 0 )</td>
</tr>
<tr>
<td>jth variable ( \leq 0 )</td>
<td>jth constraint ( \leq )</td>
</tr>
<tr>
<td>jth variable free</td>
<td>jth constraint ( = )</td>
</tr>
<tr>
<td>jth variable ( \geq 0 )</td>
<td>jth constraint ( \geq )</td>
</tr>
</tbody>
</table>

1.2 Shadow Prices

The dual formulation creates a new set of variables, \( y \). Each variable \( y \) is associated with a corresponding constraint from the primal problem. In particular, at the dual optimal solution, the value of the \( y \) variables tells us how much the objective will increase by relaxing a given constraint by 1 unit. Note that the full value of this increase may not be attainable if another constraint binds first – rather the shadow price pertains to a small \( \epsilon \)-distance from the current value of the constraint RHS.
2 The Primal-Dual Relationship

2.1 Duality Gap

There is a strong relationship between the objective values of the primal and the dual program. In Figure 1, we represent the possible objective values of both programs. We might logically ask if a gap can exist between the two objective values (and if this gap can in fact be negative, which would be the case if the optimal objective values could cross each other).

2.2 Weak Duality Theorem

The following theorem gives an answer to this question.

**Theorem 2.1** (Weak Duality Theorem). The (primal) objective value of any feasible primal solution is no larger than the (dual) objective value of any feasible dual solution:

\[
\text{Primal Optimal Value} \leq \text{Dual Optimal Value}
\]

A proof of this theorem is given in the lecture notes. While this result may be called 'weak', it can be used in powerful ways, as we shall see next.

2.3 Certificate of Optimality

It is quite easy to prove the following corollary of the Weak Duality Theorem (again see the lecture notes for the actual treatment):
Corollary 2.1 (Certificate of Optimality). Given primal feasible solution $x^*$ and dual feasible solution $y^*$ where the primal and dual objectives are equal, then $x^*$ is optimal for the primal problem and $y^*$ is optimal for the dual problem.

That independent of any algorithm for solving LP’s one can verify if a given primal-dual solution pair is in fact optimal. All you need to do is verify that both solutions are feasible, and that the resulting objective values in the two problems are identical.

2.4 Strong Duality Theorem

It turns out that we can actually prove the much stronger result, that no gap exists between the values of optimal primal and dual solutions:

Theorem 2.2 (Strong Duality Theorem). If an LP has an optimal solution then so does its dual and the two optimal values are equal.

Again, refer to the lecture notes for the proof (which appeals to Simplex). The lemma used to prove this result has a practical implication: it gives us a way to read off the dual solution from the final simplex tableau we see when solving the primal. This is probably best illustrated with an example.

Exercise 1

We known that an LP has to be either (a) infeasible, (b) has optimal solution, or (c) unbounded. Based on the duality theorems, answer the following questions.

1. If the primal LP is unbounded, which of (a), (b), and (c) are possible for the dual?
2. If the primal LP has optimal solution, which of (a), (b), and (c) are possible for the dual?
3. If the primal LP is infeasible, which of (a), (b), and (c) are possible for the dual?

End Exercise 1

2.5 Converting from Primal to Dual: An Example

Suppose we begin with the following example:

$$\begin{align*}
\text{max } & 2x_1 + 3x_2 - x_3 \\
\text{s.t. } & x_1 + x_2 \leq 8 \\
& x_2 - 3x_3 \leq 6 \\
& x_1, x_2, x_3 \geq 0
\end{align*}$$

First we construct the following tableau using slack variables $x_4$ and $x_5$: 

By performing simplex pivot operations, we arrive at the following optimal tableau:

\[
\begin{array}{c|ccccc}
  & x_1 & x_2 & x_3 & x_4 & x_5 \\
\hline
z & -2 & -3 & 1 & 0 & 0 & 0 \\
x_4 & 1 & 0 & 1 & 0 & 0 & 8 \\
x_5 & 0 & 1 & -3 & 0 & 1 & 6 \\
\end{array}
\]

Thus, \( z=0 \), \( x=(0,0,0,8,6) \), \( B=(4,5) \)

Since our reduced costs are all non-negative, we are done.

Exercise 2

Create the dual formulation, by following equation (1), above:

End Exercise 2

Exercise 3

By constructing a certificate of optimality, confirm that the objective value to the primal is optimal. (Hint: To do this, first compute the optimal dual solution by using \( y^T = c_B^T A_B^{-1} \). The \( c_B \) vector can be read off from the initial LP given the basis at the optimal primal tableau. The \( A_B^{-1} \) can be read off the optimal primal tableau. Then verify that the computed dual solution is feasible and the optimal objective value of the dual is the same as that of the primal LP.

End Exercise 3
By completing the above exercises, you should now:

- understand the relationship between the dual and primal linear program and the utility of the dual
- be comfortable converting the primal problem to the dual problem and vice versa.

3 Sensitivity Analysis

So far we have been happy once we found an optimal solution to an LP. Now we want to look at how “robust” this solution is to changes in our input data.

Given an LP in ‘standard equality form’:

$$\begin{align*}
\text{max} & \quad c^T x \\
\text{s.t.} & \quad Ax = b \\
& \quad x \geq 0
\end{align*}$$

we can analyze changes in the cost vector $c$, in the constraint matrix $A$ and in the RHS of the constraints, $b$. But before we do so, let’s review some of the algebra that’s necessary for such an analysis.

3.1 Algebra Review

Let’s start with an LP in standard equality form and then apply some algebra:

$$\begin{align*}
Ax &= b \\
A_Bx_B + A_{B'}x_{B'} &= b \\
A_Bx_B &= b \\
\iff x_B &= A_B^{-1}b = \bar{b}
\end{align*}$$

In equation 3, we split up $A$ and $x$ into the basic parts $(A_B, x_B)$ and the non-basic parts $(A_{B'}, x_{B'})$. Next, at any basic solution, we have equation 4 since non-basic variables are 0. Note that $A_B$ is invertible since it is a square matrix ($m$ rows and columns) and $B$ is a basis. We can
multiply the whole equation with $A_B^{-1}$, giving equation 5. Remember that we denoted the RHS of the constraint rows $\bar{b}$ to indicate that this vector results from $b$ and a series of row operations.

In lecture, we derived similar algebraic expressions for other parts of the tableau (corresponding to basis $B$) and its relationship to the dual solution. Here is a summary of all the expressions we derived:

- Basic solution: $x_B = A_B^{-1}b = \bar{b}$
- Dual solution: $y^T = c_B^T A_B^{-1}$
- Objective value: $z = c_B^T x_B = c_B^T A_B^{-1}b = y^T b$
- Basic reduced cost: $\bar{c}_j = c_B^T A_B^{-1} A_j - c_j = y^T A_j - c_j = 0$
- Non-basic reduced costs: $\bar{c}_{B'}^T = c_B^T A_B^{-1} A_{B'} - c_{B'}^T$

You should be able to verify, going through the Simplex algorithm, that these algebraic expressions in fact correspond to the elements of the simplex tableau.

### 3.2 Sensitivity: What if something changes?

When analyzing the sensitivity of a basic feasible solution to changes in the input data, we must differentiate the following 7 cases:

1. Changing objective function coefficient of a non-basic variable.
2. Changing objective function coefficient of a basic variable.
3. Changing the RHS.
4. Changing the entries in the column of a non-basic variable.
5. Changing the entries in the column of a basic variable.
6. Adding a variable to the LP.

#### Exercise 4

Let’s write down a brief summary of sensitivity analysis. For each type of change, please write down the effect on the optimal solution and the conditions for which the current basis still remains optimal.

1. Changing nonbasic objective function coefficient $c_j$
2. Changing basic objective function coefficient $c_j$
3. Changing RHS $b_i$ of a constraint
4. Changing column entries of a nonbasic variable $x_j$, or adding a new variable

End Exercise 4
For each change, a different algebraic derivation is necessary to see which conditions have to be checked to make sure optimality is maintained even after the change. Please see the lecture notes for the individual derivations. We will only look at one particular example here.

3.3 An example: Changing the objective function coefficient of a basic variable

Let’s go through one specific example. Consider the following LP:

\[
\begin{align*}
\text{maximize} & \quad 5x_1 + x_2 - 12x_3 \\
\text{subject to} & \quad 3x_1 + 2x_2 + x_3 = 10 \\
& \quad 5x_1 + 3x_2 + x_4 = 16 \\
& \quad x_1, x_2, x_3, x_4 \geq 0
\end{align*}
\]

Using the Simplex algorithm until all reduced costs are non-positive, we arrive at the following tableau (you can check by yourself!):

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>z</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>1</td>
<td>5</td>
<td>-3</td>
<td>2</td>
</tr>
<tr>
<td>$x_1$</td>
<td>1</td>
<td>0</td>
<td>-3</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$z$</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>7</td>
<td>12</td>
</tr>
</tbody>
</table>

$x = \{2, 2, 0, 0\}, z = 12$.

Note that the rows in the final tableau correspond to $x_2$ and $x_1$, in this specific order! This particular order plays a role when we want to read off information from the final tableau!!

Question: what happens when the objective function coefficient of $x_1$, i.e. $c_1$, changes by $\varepsilon$, i.e. from 5 to $5 + \varepsilon$?

3.3.1 Sensitivity Analysis by Hand

Note that $c_1$ is part of $c_B$ because $x_1$ is a basic variable. Thus, let’s look back at the algebraic expressions that we have derived to see in which expressions $c_B$ shows up.

Exercise 5

In which of the algebraic expressions does $c_B$ show up?

End Exercise 5
Thus, we have to check the reduced costs of all variables to see if the optimality conditions are still satisfied (all reduced costs non-positive).

**Basic Variables:** Recall our expression for the objective coefficients for basic variables and let’s do some algebra: \( \bar{c}_j = c_B^T A_B^{-1} A_j - c_j = c_B^T e_j - c_j = c_j - \bar{c}_j = 0 \), where \( e_j \) is the \( j \)th unit vector.

Thus, we see that the reduced costs of all basic variables stay at zero.

**Non-Basic Variables:** Recall our expression for the objective coefficients for non-basic variables: \( \bar{c}_{B'} = c_B^T A_B^{-1} A_{B'} - c_{B'} \). We now have to check the new reduced costs, \( c_3 \) and \( c_4 \).

First, note that here \( c_B^T = (5 + \varepsilon, 1) \)
Second, note that here \( A_B^{-1} = \begin{pmatrix} -3 & 2 \\ 5 & -3 \end{pmatrix} \), which we can read off from the final tableau.

**Exercise 6**

Compute the new reduced costs \( \bar{c}_3 \) and \( \bar{c}_4 \) using the algebraic expression

\[
\bar{c}_{B'} = c_B^T A_B^{-1} A_{B'} - c_{B'}
\]

**End Exercise 6**

To maintain optimality, even after \( c_1 \) has changed by \( \varepsilon \), the reduced costs must be non-negative. Thus we can derive the following conditions:

**Exercise 7**

Compute the conditions on \( \varepsilon \) such that the optimal solution does not change even if \( c_1 \) has been increased by \( \varepsilon \). (Hint: Basically, we want to ensure that the new reduced costs computed in the previous exercise are always non-negative.)

**End Exercise 7**
3.3.2 Sensitivity Analysis in AMPL/CPLEX (see sensitivity.mod)

Now that we have done an example for a sensitivity analysis by hand, let’s see how this works in AMPL. We first write down the formulation of the LP in AMPL:

```ampl
var x1 >= 0;
var x2 >= 0;
var x3 >= 0;
var x4 >= 0;

maximize objective: 5*x1 + x2 -12*x3;
subject to constraint1: 3*x1 + 2*x2 + x3 = 10;
subject to constraint2: 5*x1 + 3*x2 + x4 = 16;
```

We then solve the model, setting the solves as CPLEX and telling CPLEX to provide sensitivity information and use the Simplex algorithm:

```ampl
model sensitivity4.mod;
option solver cplexamp;
option cplex_options 'sensitivity primalopt';
solve;
```

We obtain the sensitivity report by querying the bounds on the cost coefficient for \(x_1\):

```ampl
ampl: display x1.current; x1.current = 5
ampl: display x1.up; x1.up = 5.66667
ampl: display x1.down; x1.down = 1.5
```

which agrees with our findings that \(c_1\) can increase by at most \(\frac{2}{3}\) and decrease by at most 3.5.

Note, the bounds on the RHS constraints (which we have not analyzed by hand!) could be displayed with the following commands:
ampl: display constraint1.current; constraint1.current = 10
ampl: display constraint1.up; constraint1.up = 10.6667
ampl: display constraint1.down; constraint1.down = 9.6

4 Understanding Sensitivity Analysis

4.1 Remember from Lecture

- Basic solution: \( x_B = A_B^{-1}b = \bar{b} \)
- Dual solution: \( y^T = c_B^T A_B^{-1} \)
- Objective value: \( z = c_B^T x_B = c_B^T A_B^{-1} b = y^T b \)
- Non-basic reduced cost: \( \bar{c}_B' = c_B^T A_B^{-1}A'_B - c_B' \)
- Basic reduced cost: \( \bar{c}_j = c_B^T A_B^{-1}A_j - c_j = y^T A_j - c_j \)

4.2 Let's get some Practice

Consider the following LP:

\[
\begin{align*}
\text{maximize} & \quad 5x_1 + x_2 - 12x_3 \\
\text{subject to} & \quad 3x_1 + 2x_2 + x_3 = 10 \\
& \quad 5x_1 + 3x_2 + x_4 = 16 \\
& \quad x_1, x_2, x_3, x_4 \geq 0
\end{align*}
\]

Solving with AMPL/CPLEX, we obtain the following solution for this LP:

\[ x = (2, 2, 0, 0) \quad \text{and} \quad z = 12 \quad (6) \]

Furthermore, let’s assume we get the following sensitivity analysis output from AMPL (see sensitivity.mod and sensitivit.run):

\[
\begin{array}{cccccc}
& \text{varname} & \text{var.rc} & \text{var.down} & \text{var.current} & \text{var.up} \\
1 & x1 & -7.10543e-15 & 1.5 & 5 & 5.66667 \\
2 & x2 & -3.55271e-15 & 0.6 & 1 & 3.33333 \\
3 & x3 & -2 & -1e+20 & -12 & -10 \\
4 & x4 & -7 & -1e+20 & 0 & 7 \\
\end{array}
\]

\[
\begin{array}{cccccc}
& \text{conname} & \text{con.dual} & \text{con.down} & \text{con.current} & \text{con.up} \\
1 & \text{constraint1} & -10 & 9.6 & 10 & 10.6667 \\
2 & \text{constraint2} & 7 & 15 & 16 & 16.6667 \\
\end{array}
\]
Exercise 8
For the following questions, either answer them using the information from the sensitivity analysis or state if you need more information to answer it.

1. Assume the the RHS of constraint 1 increases by 0.5. Is the current basis still optimal? What happens to the optimal value?
2. What happens to the optimal value if the RHS of constraint 2 decreases by 1?
3. What happens to the optimal value if the RHS of constraint 2 increases by 1?

End Exercise 8

Exercise 9
For the following questions, either answer them using the information from the sensitivity analysis or state if you need more information to answer it.

1. Assume we increase the objective coefficient of variable $x_1$ by 0.5. Is the current basis still optimal? What happens to the optimal value?
2. Assume we increase the objective coefficient of variable $x_4$ by 3. Is the current basis still optimal? What happens to the optimal value?
3. Assume we increase the objective coefficient of variable $x_3$ by 3. Is the current basis still optimal? What happens to the optimal value?

End Exercise 9
By completing the above exercises, you should now:

- understand how robustness of a solution can be analyzed using sensitivity analysis and be able to conduct sensitivity analysis by hand.

- understand sensitivity information in AMPL/CPLEX and be comfortable determining how changes in the input affect the optimal solution using sensitivity analysis output from AMPL.

5 The Dual Simplex Algorithm

5.1 Review

We have already seen how we can use duality to check optimality of a basic solution at hand. Now, we will see a very practical algorithm, called the Dual Simplex Algorithm, that uses duality and can be applied in situations where the usual simplex algorithm, primal simplex, cannot.

We will see in lecture that there is a direct correspondence between variables in the primal simplex algorithm and the variables in the dual simplex algorithm. In particular:

1. For every primal tableau, there is a corresponding dual tableau

2. A primal variable $x_k$ is basic as long as the corresponding dual variable $y_k$ is non-basic.

3. Pivots in the dual tableau correspond to pivots in the primal tableau

4. Primal $\bar{b} \geq 0 \iff$ dual $\bar{c} \geq 0$
5. Primal $\bar{c} \geq 0 \Leftrightarrow$ dual $\bar{b} \geq 0$

6. Remember: A tableau is primal feasible as long as the RHS, i.e. $\bar{b}$, is non-negative.

7. Definition: A tableau is dual feasible as long as the reduced costs, i.e. $\bar{c}$, are non-negative.

8. Remember: Primal optimality is reached when we have found a primal feasible solution and all reduced costs are non-negative.

9. Definition: Dual optimality is reached when we have found a dual feasible solution and the RHS, i.e. $\bar{b}$, is non-negative.

The **dual simplex algorithm**, which maintains primal optimality (dual feasibility) while working toward primal feasibility (dual optimality), is defined as follows:

- Assume you have a dual feasible tableau ($\bar{c} \geq 0$).

- **Step 1:** Pick a basic variable $x_r$ to leave with a strictly negative RHS, i.e. $\bar{b}_r < 0$. If no such variable exists you found an optimal solution. STOP.

- **Step 2:** Pick a nonbasic variable $x_k$ to enter by considering row $r$ and strictly negative (nonbasic) entries. Ratio test: $k$ should satisfy $-\bar{c}_k/\bar{a}_{rk} = \min\{-\bar{c}_j/\bar{a}_{rj} : j \notin B, \bar{a}_{rj} < 0\}$ If all $\bar{a}_{rj}$ from $j \notin B$ are nonnegative, then dual unbounded and primal infeasible and STOP.

- **Step 3:** Pivot on $(r,k)$ and go to Step 1.

5.2 Practice

Now that we have described the dual simplex algorithm, let’s get some practice applying it. Let’s consider the LP from the beginning again:

$$\text{maximize} \quad 5x_1 + x_2 - 12x_3$$

$$\text{subject to} \quad 3x_1 + 2x_2 + x_3 = 10$$

$$5x_1 + 3x_2 + x_4 = 16$$

$x_1, x_2, x_3, x_4 \geq 0$

Using the Simplex algorithm until all reduced costs are non-negative, we arrive at the following tableau (you can check by yourself!):

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z$</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0</td>
<td>1</td>
<td>5</td>
<td>-3</td>
</tr>
<tr>
<td>$x_1$</td>
<td>1</td>
<td>0</td>
<td>-3</td>
<td>2</td>
</tr>
</tbody>
</table>

$x = \{2, 2, 0, 0\}, z = 12$. 

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Exercise 10

For the following exercises, remember the algebraic relations listed at the beginning of the section notes and refer to the sensitivity analysis from exercise 1 above. Remember from the sensitivity analysis that when the RHS of constraint 2 is increased by 1, we are outside the allowable range and the optimal basis changes. We can use the dual simplex algorithm to proceed:

1. Assume, the RHS of the constraint 2 is increased by 1. Compute the new RHS, i.e. \( \bar{b} \).
2. Compute the new objective value \( z \) for the (dual feasible) tableau.
3. Construct the new tableau corresponding to the current basis but considering the change in \( b \).
4. By performing one iteration of the dual simplex algorithm, we arrive at the following tableau:

\[
\begin{array}{cccc|c}
  \quad & x_1 & x_2 & x_3 & x_4 \\
  z & 0 & \frac{7}{3} & 13\frac{2}{3} & 0 & 16\frac{2}{3} \\
 x_4 & 0 & -\frac{2}{3} & -\frac{2}{3} & 1 & \frac{1}{3} \\
 x_1 & 1 & \frac{2}{3} & \frac{1}{3} & 0 & 3\frac{1}{3}
\end{array}
\]

Explain what are the entering and leaving variables, why are they chosen, and what is the pivot operation? Explain why the dual simplex algorithm is finished. Given the optimal solution and its objective value.

End Exercise 10
By completing the above exercises, you should now:

- understand how the dual simplex algorithm works.
- be comfortable applying the dual simplex algorithm to solve an LP.

6 Complementary Slackness

6.1 Review

There is a strong correspondence between primal and dual problems. Among these, we have seen that the constraints in one problem are complementary to the variables in the other. Stemming from this, complementary slackness refers to an association between the slack in primal inequality constraints and positivity of their corresponding dual variable. More formally:

Theorem 6.1 (Complementary Slackness). Consider primal problem $P$ in standard inequality form and corresponding dual problem $D$. If the primal problem $(P)$ has solution $x^*$ and the dual problem $(D)$ has solution $y^*$, then:

- If $x_j^* > 0$, then the $j$th constraint in $(D)$ is binding.
- If the $j$th constraint in $(D)$ is not binding, then $x_j^* = 0$.
- If $y_i^* > 0$, then the $i$th constraint in $(P)$ is binding.
- If the $i$th constraint in $(P)$ is not binding, then $y_i^* = 0$.

\footnote{Content from this section based on http://www.econ.ucsd.edu/jsobel/172aw02/notes6.pdf}
This says that if a variable is positive, then the associated dual constraint must be binding. Further it says if a constraint fails to bind, then the associated dual variable must be zero. In other words, there cannot be 'slackness' in both a variable and its dual constraint, or a constraint and its dual variable. However note that it is possible for there to be no 'slackness' in both places: a binding constraint with a associated 0 dual variable.

Now we can offer a sketch as to why these conditions hold:

**Proof** (Complementary Slackness). Suppose we begin with a primal and dual in standard inequality form:

\[
\begin{align*}
\text{max } & \ c^T x \\
\text{Ax } & \leq b \\
x & \geq 0
\end{align*}
\]

and

\[
\begin{align*}
\text{min } & \ b^T y \\
A^T y & \geq c \\
y & \geq 0
\end{align*}
\]

Consider a feasible solution to the primal LP denoted by \( x \) and a feasible solution to the dual LP denoted by \( y \), then we have

\[
c^T x \leq (A^T y)^T x \leq y^T A x \leq y^T b = b^T y
\]

If \( x^* \) solves the primal and \( y^* \) solves the dual, then this becomes:

\[
\begin{align*}
\text{cx}^* &= y^* A x^* = b y^* \\
\Rightarrow & \ (c - y^* A)x^* = 0 \\
\Rightarrow & \ \sum_{j=1}^n (c - y^* A)_j x^*_j = 0 \\
\Rightarrow & \ (c - y^* A)_j x^*_j = 0, \forall j \in \{1, ..., n\}
\end{align*}
\]

Because \( y^* \) is feasible for the dual we know that \((c - y^* A)_j \) must be non-negative. Further, because \( x^*_j \) is feasible for the primal, it must also be non-negative. Multiplying must then give us a non-negative number. And, if the sum of these non-negative numbers is zero, each one in fact must be zero. Thus we have \((c - y^* A)_j x^*_j = 0, \forall j \in \{1, ..., n\}\). But this can only happen if either or both of \((c - y^* A)_j \) and \( x^*_j \) is zero. And this is precisely another way of saying the Complementary Slackness condition: if \( x^*_j > 0 \), then the \( j \)th dual constraint binds, i.e. \((c - y^* A)_j = 0 \). Moreover, if the \( j \)th dual constraint does not bind, i.e. \((c - y^* A)_j \geq 0 \), then \( x^*_j \) must be zero. The other Complementary Slackness conditions follow from identical reasoning about the second inequality in (7).

Complementary Slackness is useful not only because it helps us to interpret the meaning of dual variables, but also because it provides a convenient way to find the dual optimal solution knowing the primal optimal solution.

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6.2 An example

Consider the diet problem which is formulated as follows: We want to minimize the amount spent on food subject to meeting nutritional constraints. The given information for the problem are the costs of food, the nutritional requirements, and the nutrient content in each of the foods. For this problem, we can also formulate the dual. Here the problem is to find prices of nutrient pills to maximize the cost of the nutrients needed subject to the constraint that nutrients supplied in any food can be purchased more cheaply by buying pills than by buying the food. Let us think about the solution to the dual as providing prices of nutrients - in other words they represent how much the food buyer is willing to pay to get the nutrients directly (instead of in the food).

Consider what the complementary slackness conditions mean in this context. Suppose that when you solve the diet problem you find that it is part of the solution to buy a positive amount of the first food $x_1$. The complementary slackness condition then implies that the first dual constraint must be binding. That means that the price of (one unit of) the first food is exactly equal to the cost of (one unit of) the nutrients supplied by the food. Why should this be true? Imagine that the pill seller is offering nutrients at the prices that solve the dual. If you buy a positive amount of the first food, then it must be no more expensive to do that than to get the nutrients through pills. What if the first dual constraint is not binding? The first dual constraint states that the price of one unit of the first food should be at least as great as the cost of the nutrients inside the food. If the constraint does not bind, then it means that the food is actually more expensive than the nutrients inside it. That is, the food is strictly more expensive than its nutrient content. If this happens, then it makes no sense to buy the food: $x_1^* = 0$.

If a constraint in the primal does not bind, then there is some nutrient that it oversupplied when you solve the diet problem. If so how much would you be willing to pay for a pill containing the oversupplied nutrient? Complementary Slackness says that the answer is zero. If your food supplies more of a nutrient than you need, then the price of that nutrient - that is, how much you are willing to pay for the nutrient - is zero. Now suppose that a dual variable is positive. In that case, you are willing to pay a strictly positive amount for a nutrient supplied by the pill. Complementary Slackness says that (at a solution) it must be the case that you are getting exactly the amount of the nutrient you need (not extra). The complementary slackness conditions guarantee that the values of the primal and dual are the same. In the diet problem, the pill seller guarantees that pills are no more expensive than food. Complementary Slackness guarantees that when you solve the problem, pills cost exactly the same as food for those foods that you actually buy.

Let us try to write an LP for the diet problem with some data. Suppose that we want to consume 2 foods: Food1 costs $10 per unit and Food2 costs $15 per unit. Each food contains 2 nutrients: vitamin A and vitamin B. Food1 contains 3 units of vitamin A and 5 units of vitamin B. Food2 contains 2 units of vitamin A and 7 units of vitamin B. The total daily nutrient requirement is 22 units of vitamin A and 37 units of vitamin B.

We will formulate the primal problem as one of minimizing the cost of purchasing the 2 foods. Let $f_1$ and $f_2$ denote the number of units of Food1 and Food2 respectively. Then we get the following LP:
minimize $10f_1 + 15f_2$
subject to $3f_1 + 2f_2 \geq 22$
$5f_1 + 7f_2 \geq 37$
$f_1, f_2 \geq 0$

The dual problem can be formulated as one of maximizing the prices of the nutrients. Let $p_1$ and $p_2$ denote the number of units of vitamin A and vitamin B respectively. Then we get the following LP:

maximize $22p_1 + 37p_2$
subject to $3p_1 + 5p_2 \leq 10$
$2p_1 + 7p_2 \leq 15$
$p_1, p_2 \geq 0$

6.3 Practice

Suppose we have the following problem:

max $2x_1 + 4x_2 + 3x_3 + x_4$
s.t. $3x_1 + x_2 + x_3 + 4x_4 \leq 12$
$x_1 - 3x_2 + 2x_3 + 3x_4 \leq 7$
$2x_1 + x_2 + 3x_3 - x_4 \leq 10$
$x_1, x_2, x_3, x_4 \geq 0$

If we solve this problem using Simplex, we will find that Optimal Value of 42 can be obtained at the solution vector $(0, 10.4, 0, .4)$.

Now, we will use the complementary slackness conditions to solve for the dual optimal solution without using the simplex algorithm.

Exercise 11

From the optimal basic solution $x = (0, 10.4, 0, .4)$, we know that $x_2$ and $x_4$ are positive in the optimal solution. What does this imply about the 2nd and 4th dual constraints based on the complementary slackness conditions? The dual is as follows:

min $12y_1 + 7y_2 + 10y_3$
s.t. $3y_1 + y_2 + 2y_3 \geq 2$
$y_1 - 3y_2 + y_3 \geq 4$
$y_1 + 2y_2 + 3y_3 \geq 3$
$4y_1 + 3y_2 - y_3 \geq 1$
$y_1, y_2, y_3 \geq 0$

End Exercise 11
Exercise 12

Note that at our primal solution, the first and third primal constraints bind. But the second one does not – there is slack in this constraint:

\[ x_1 - 3x_2 + 2x_3 + 3x_4 \leq 7 \]
\[ 1(0) - 3(10.4) + 2(0) + 3(4) \leq 7 \]
\[ -30 \leq 7 \]

What does this imply about the 2nd dual variable \( y_2 \)?

End Exercise 12

Exercise 13

Solve the system of equations to find the dual optimal solution. What is the optimal objective value?

End Exercise 13
By completing the above exercises, you should now:

- understand the concept of complementary slackness.
- be comfortable using complementary slackness to solve the dual LP.
7 Solutions

Solution 1
1. If the primal LP is unbounded, the dual LP must be infeasible.

2. If the primal LP has optimal solution, the dual LP must have an optimal solution with the same optimal objective value.

3. If the primal LP is infeasible, the dual LP could either be (a) infeasible or (c) unbounded.

End Solution 1

Solution 2

\[
\begin{align*}
\min \quad & 8y_1 + 6y_2 \\
\text{s.t.} \quad & y_1 \geq 2 \\
& y_1 + y_2 \geq 3 \\
& -3y_2 \geq -1 \\
& y_1, y_2 \geq 0
\end{align*}
\]

End Solution 2

Solution 3
As shown in the lecture notes, the optimal solution to this dual will be \( y^T = c_B^T A_B^{-1} \), where \( B \) is the basis from the optimal tableau above. Since this basis is (3,2), our \( c_B \) vector will be \([-1,3]\). Further, it turns out that the \( A_B^{-1} \) matrix can be read as the columns of the original basis (in this case based on the slack variables) from the final tableau. Reading this off our final tableau we get:

\[
\begin{bmatrix}
\frac{1}{3} & -\frac{1}{3} \\
1 & 0
\end{bmatrix}
\]

Multiplying these together, we get \([2, \frac{1}{3}]\). If you substitute this back into the objective function of the dual formulation, you arrive at an objective value of \(23\frac{1}{3}\). Further you can verify that all three constraints are satisfied by this solution. Thus because we have a feasible solution to both the primal and the dual, and these solutions have the same objective value, we have a certificate of optimality that we do indeed have the optimal solution.

End Solution 3
Solution 4

1. Changing nonbasic objective function coefficient $c_j$
   (a) Reduced cost $\bar{c}_j$ is changed
   (b) Need reduced cost $\bar{c}_j \geq 0$

2. Changing basic objective function coefficient $c_j$
   (a) All reduced costs may change, objective value may change
   (b) Need reduced cost $\bar{c}_i \geq 0$ for all nonbasic variables $x_i$

3. Changing RHS $b_i$ of a constraint
   (a) RHS of constraints and also objective value may change
   (b) Need RHS $\bar{b}_i \geq 0$ for each constraint

4. Changing column entries of a nonbasic variable $x_j$ or adding a new variable $x_j$
   (a) Changes reduced cost on $x_j$ and also the constraint column $A_j$
   (b) Need reduced cost $\bar{c}_j \geq 0$

End Solution 4

Solution 5

- It does not show up in the RHS, $\bar{b} = A_B^{-1}b = x_B$. Thus, we don’t have to worry about feasibility.

- It does show up in the expressions for the reduced costs for both basic and non-basic variables.
  - Basic reduced cost: $\bar{c}_j = c_B^T A_B^{-1} A_j - c_j = y^T A_j - c_j$
  - Non-basic reduced cost: $\bar{c}_{B'}^T = c_B^T A_B^{-1} A_{B'} - c_{B'}^T$

End Solution 5
Solution 6

First, we compute the dual solution \( y^T = c_B^T A_B^{-1} \) as follows:

\[
y^T = c_B^T A_B^{-1} \\
= (5 + \varepsilon, 1) \begin{pmatrix} -3 & 2 \\ 5 & -3 \end{pmatrix} \\
= (-15 - 3\varepsilon + 5, 10 + 2\varepsilon - 3) \\
= (-10 - 3\varepsilon, 7 + 2\varepsilon)
\]

Now, we can use our algebraic expression for the reduced costs \( \bar{c}_3 \) and \( \bar{c}_4 \) and compute those. Thus, we get:

\[
\bar{c}_3 = y^T A_3 - c_3 \\
= (-10 - 3\varepsilon, 7 + 2\varepsilon) \cdot (1, 0)^T + 12 \\
= -10 - 3\varepsilon + 12 \\
= 2 - 3\varepsilon
\]

and

\[
\bar{c}_4 = y^T A_4 - c_4 \\
= (-10 - 3\varepsilon, 7 + 2\varepsilon) \cdot (0, 1)^T - 0 \\
= +7 + 2\varepsilon - 0 \\
= 7 + 2\varepsilon
\]

End Solution 6
Solution 7

\[ c_3 \geq 0 \]
\[ \iff 2 - 3\varepsilon \geq 0 \]
\[ \iff 2 \geq 3\varepsilon \]
\[ \iff \varepsilon \leq \frac{2}{3} \]

and

\[ c_4 \geq 0 \]
\[ \iff +7 + 2\varepsilon \geq 0 \]
\[ \iff 2\varepsilon \geq -7 \]
\[ \iff \varepsilon \geq -\frac{7}{2} \]

Putting the two together, we get:

\[ -\frac{7}{2} \leq \varepsilon \leq \frac{2}{3} \]

which completes the sensitivity analysis for the objective coefficient of \( x_1 \).

End Solution 7

Solution 8

1. Looking at the sensitivity analysis for constraint 1, we see that an increase of 0.5 is within the allowable range of change (+2/3 is OK). Thus, the basis will not change.

Now, recall the equation for the objective value: \( z = y^Tb \). Thus, we only have to look at the corresponding dual variables in the sensitivity analysis to compute the change of the objective value. The total cost changes by \( 0.5 \cdot (-10) = -5 \). Thus, the new cost is 7.

2. This change is also in the allowable range of change. Thus, looking at the dual variables we know that the total cost changes by \(-1 \cdot 7 = -7\). Thus, the new cost is 5.

3. Looking at the sensitivity analysis, we see that an increase of 1 is outside of the allowable range and the optimal basis will change. Thus, we cannot answer this question directly with the information present.

End Solution 8
Solution 9

1. By looking at the sensitivity analysis for variable $x_1$ we see that an increase of 0.5 is within the allowable range of change ($x_1$ can be between 1.5 and 5.6666667). Thus, the current basis remains optimal. The new optimal value changes by $0.5 \cdot 2 = 1$. Thus the new total value is 13.

2. The cost increase is within the allowable range for variable $x_4$. Thus, the current basis remains optimal. Because $x_4$ is a non-basic variable, i.e. it is set equal to 0, this change in the cost coefficient does not change the objective value.

3. This cost increase is outside of the allowable range for variable $x_3$ (var.up was -10). Thus, the optimal basis will change and we cannot answer this question fully without doing more computation.

End Solution 9

Solution 10

1. We can read off $A_B^{-1}$ from the final tableau, i.e. $A_B^{-1} = \begin{pmatrix} -3 & 2 \\ 5 & -3 \end{pmatrix}$. We know that

\[ \bar{b} = A_B^{-1}b, \text{ thus } \bar{b} = \begin{pmatrix} -3 & 2 \\ 5 & -3 \end{pmatrix} \cdot \begin{pmatrix} 10 \\ 17 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}. \]

2. From the AMPL output for the sensitivity analysis we know that the dual variables are: $y = (-10, 7)$. Thus, $z = y^T b = (-10, 7) \cdot \begin{pmatrix} 10 \\ 17 \end{pmatrix} = 19$.

3. Thus, the new tableau looks as follows:

\[
\begin{array}{c|cccc|c}
   & x_1 & x_2 & x_3 & x_4 & z \\
---&---&---&---&---&---
x_1 & 0 & 0 & 2 & 7 & 19 \\
x_2 & 0 & 1 & 5 & -3 & -1 \\
x_1 & 1 & 0 & -3 & 2 & 4
\end{array}
\]

4. Because $x_2$ is now negative, it has to leave the basis. Thus we pivot on row 1. The only negative entry in this row correspond to $x_4$, thus $x_4$ will enter the basis.

The new tableau is dual optimal. Thus, we are done. We found a new basic feasible solution $x = (3\frac{1}{3}, 0, 0, \frac{1}{3})$ with optimal objective value $z = 16\frac{2}{3}$.

End Solution 10
Solution 11
What do the complementary slackness conditions tell us? They tell us the dual constraints for positive primal variables will bind, and thus in fact be satisfied as equalities. Because we know $x_2$ and $x_4$ are positive, we 2nd and 4th dual constraints should be binding (equalities):

\[
\begin{align*}
y_1 - 3y_2 + y_3 & = 4 \\
4y_1 + 3y_2 - y_3 & = 1
\end{align*}
\]

End Solution 11

Solution 12
The second primal constraint has a slack of $7 - (-30) = 37 > 0$. Complementary slackness then implies that the corresponding dual variable, $y_2$ should be 0. Giving us the following set of equations:

\[
\begin{align*}
y_1 - 3y_2 + y_3 & = 4 \\
4y_1 + 3y_2 - y_3 & = 1 \\
y_2 & = 0
\end{align*}
\]

Or:

\[
\begin{align*}
y_1 + y_3 & = 4 \\
4y_1 - y_3 & = 1 \\
y_2 & = 0
\end{align*}
\]

End Solution 12

Solution 13
The system of equations

\[
\begin{align*}
y_1 + y_3 & = 4 \\
4y_1 - y_3 & = 1 \\
y_2 & = 0
\end{align*}
\]

tells us that $y_1 = 1$ and $y_3 = 3$. This gives us a dual solution of $y = (1, 0, 3)$. We can check that this solution is feasible for the other dual constraints (it is). And we can check that the value of this solution in the dual problem is 42, which is equal to the value of our primal solution and thus optimal.

End Solution 13