Problem 1

Consider the two-player game given by the following matrix. (A positive payoff goes to the row player.)

\[
\begin{pmatrix}
4 & 1 & 0 & -3 \\
6 & -3 & -2 & 0 \\
-3 & -2 & 5 & -3 \\
-4 & 4 & -5 & 5 \\
\end{pmatrix}
\]

(a) (4 points) Write down the linear program to determine the row player strategy that maximizes the value of the game to the row player. Do the same for the column player.

(b) (2 points) Find an LP solver and use it to solve these linear programs, and give the proper strategies for both players. There are many online you can use, but we used [http://www.phpsimplex.com/simplex/simplex.htm?l=en](http://www.phpsimplex.com/simplex/simplex.htm?l=en). Note that on many solvers, all decision variables are assumed to be positive - make sure your variables respect this assumption when plugging them into the solver! (You could also use the simplex code we’re giving you for the programming problem.)

(c) (4 points) What is the value of the game? Should the column player pay the row player to play, or vice versa, and how much should one player pay the other to make the game fair?

Tip: We recommend reviewing the Section 11 notes before working on this problem.

Problem 2

Recall the SetCover problem: the solution to \textsf{SetCover}(S, n) is the minimum number of sets in the collection \( S = \{S_1, \ldots, S_m\} \) that must be selected so that their union covers \( \{1, \ldots, n\} \). Provide an algorithm taking time \( \text{poly}(n, m) \) to find a \( k \)-approximation to \textsf{SetCover}, where \( k \) is defined as \( \max_{i \in \{1, \ldots, n\}} |\{j : i \in S_j\}| \). That is, \( k \) is the maximum number of times a universe element appears in any set. \textbf{Hint:} first think about the case that \( k = 2 \).
Problem 3

A transitive reduction of a directed graph is a subgraph (i.e. same vertex set but with a subset of edges) with as few edges as possible that has the same reachability relation as the given graph. Formally, a directed graph $G'$ is said to be a transitive reduction of a directed graph $G$ if they have the same vertex sets and:

1. For every two vertices $u, v$, there is a path from $u$ to $v$ in $G'$ iff there is a path from $u$ to $v$ in $G$, and
2. $G'$ has the minimum number of edges among all graphs satisfying conditions 1 and 3.
3. $G'$ is a subgraph of $G$, i.e. if $G' = (V', E')$ and $G = (V, E)$ are the sets of vertices edges of $G'$ and $G$, then $E' \subseteq E$.

Here is an example of a graph and its transitive reduction:

![Graph and Transitive Reduction Diagram]

Show that it is NP-complete to determine, given a directed graph $G$ and a number $k$, whether $G$ has a transitive reduction with at most $k$ edges. The input $G$ need not be acyclic. You may use that the directed STRONGLYCONNECTEDHAMILTONIANCYCLE problem is NP-hard. The STRONGLYCONNECTEDHAMILTONIANCYCLE problem is defined as follows: given as input a strongly connected directed graph $G$, decide whether $G$ has a simple cycle of length $n$, where a simple cycle is defined to be a cycle with no repeated vertices.

Problem 4: Programming Problem

Solve FLOWNETWORK on the programming server (https://cs124.seas.harvard.edu). This problem is a linear programming problem, though you are not expected to write your own code to solve linear programs, as we have not covered such algorithms in class. Instead, you are free to use the simplex code we’ve provided for you in C, C++, Python, and Java at https://canvas.harvard.edu/courses/36818/discussion_topics/308716. Your main task is to model the problem as a linear program to then feed into the simplex code given.