Problem 1

There are \( n \) hungry amphibians sitting around a pond, when \( n \) insects suddenly fly overhead. Each amphibian looks at the insects and decides on a subset of them that it is willing to eat. Suppose that for any subset \( U \) of the amphibians, the collective set of insects they are willing to eat, denoted as \( N(U) \), satisfies the condition \( |N(U)| \geq |U| \). Prove that there is a way for every amphibian to eat exactly one desired insect without conflicts. **Hint:** Construct a graph and reason about flows.

Problem 2

You have been given a square plot of land that has been divided into \( n \) rows and columns, yielding \( n^2 \) square subplots. Some of these subplots have rocky ground and cannot support plant growth, while others have soil. You would like to plant palm trees on a subset of the square subplots so that every row and every column has exactly the same number \( p \) of palm trees. Furthermore, you would like to do this so that \( p \) is as large as possible.

(a) (10 points) Give an algorithm running in time \( O(n^5) \) to compute the optimal number \( p^* \) of palm trees that can be planted.

(b) (10 points) Show how to achieve an improved runtime of \( O(n^3p^*) \).

Problem 3

Consider the two-player game given by the following matrix. (A positive payoff goes to the row player.)

\[
\begin{bmatrix}
4 & 1 & 0 & -3 \\
6 & -3 & -2 & 0 \\
-3 & -2 & 5 & -3 \\
-4 & 4 & -5 & 5 \\
\end{bmatrix}
\]
(a) (8 points) Write down the linear program to determine the row player strategy that maximizes the value of the game to the row player. Do the same for the column player.

(b) (4 points) Find an LP solver and use it to solve these linear programs, and give the proper strategies for both players. There are many online you can use, but we used http://www.phpsimplex.com/simplex/simplex.htm?l=en. Note that on many solvers, all decision variables are assumed to be positive - make sure your variables respect this assumption when plugging them into the solver!

(c) (8 points) What is the value of the game? Should the column player pay the row player to play, or vice versa, and how much should one player pay the other to make the game fair?

**Programming Problem**

Solve SEPARATION on the programming server (https://cs124.seas.harvard.edu).