parametric surfaces

- define a surface $x(u^1, u^2)$ in R^3, over some patch of the U domain in R^2.
- if all goes well,

$$\frac{\partial x}{\partial u^i}$$

  are two lin ind tangent vectors in R^3

  - super/sub scripts (i..n) will represent indices.

- they form a basis for the tangent plane at $x(u)$.
  - tp: union of all tangents of curves in the surface at the point.
- fancy terminology: drop the "x"

$$\frac{\partial}{\partial u^i}$$

- even fancier: add a superscript (a..h) to tell us that it is a tangent vector

$$(\frac{\partial}{\partial u^i})^a$$

- any tangent vector can be written as

$$v^a = \sum_i (\frac{\partial}{\partial u^i})^a v^i$$

normal

- we can compute the (non unit) normal vector as

$$n = \frac{\partial x}{\partial u^1} \times \frac{\partial x}{\partial u^2}$$

change of parametrization

- suppose i have a one to one mapping from a 2d patch $(\hat{u}^{\hat{1}}, \hat{u}^{\hat{2}})$ to the 2d patch (u^1, u^2).
- then the same surface can be parametrized as

$$x(\hat{u}^{\hat{1}}, \hat{u}^{\hat{2}}) = x(u^1(\hat{u}^{\hat{1}}, \hat{u}^{\hat{2}}), u^2(\hat{u}^{\hat{1}}, \hat{u}^{\hat{2}}))$$

- at every point then there is a Jacobian (matrix) of this mapping

$$J^i_{\hat{j}} = \frac{\partial u^i}{\partial \hat{u}^{\hat{j}}}$$

- (i is row, j is column)
- we will assume J is be non-singular.

cob

- and we also have the tangent basis

$$(\frac{\partial}{\partial \hat{u}^{\hat{j}}})^a$$

- we can write the relationship between our two tangent bases as

$$(\frac{\partial}{\partial \hat{u}^{\hat{j}}})^a = \sum_i (\frac{\partial}{\partial u^i})^a J^i_{\hat{j}}$$

- just use the chain rule
- thus we can change the coordinates of a vector as

$$\begin{aligned} v^a = \sum_j (\frac{\partial}{\partial \hat{u}^{\hat{j}}})^a \hat{v}^{\hat{j}} &= \sum_j \sum_i (\frac{\partial}{\partial u^i})^a J^i_{\hat{j}} \hat{v}^{\hat{j}} \\ &= \sum_i (\frac{\partial}{\partial u^i})^a \sum_j J^i_{\hat{j}} \hat{v}^{\hat{j}} \\ &=: \sum_i (\frac{\partial}{\partial u^i})^a v^i \end{aligned}$$

- so J can be used to push forward tangent coordinates wrt "$\hat{u}$" to coordinates wrt "u"

$$v^i = \sum_j J^i_{\hat{j}} \hat{v}^{\hat{j}}$$

  - the hats in the J symbol remind us this.

covectors

- covector: an operator that takes in vectors and gives out numbers in a linear fashion is called a covector w_a.
- i apply a covector to a vector by writing them next to each other (in any order) using a shared "script: $v^a w_a$.
- lets calculate in a basis

$$\begin{aligned} v^a w_a &= w_a \sum_i (\frac{\partial}{\partial u^i})^a v^i \\ &= \sum_i w_a (\frac{\partial}{\partial u^i})^a v^i \\ &=: \sum_i w_i v^i \end{aligned}$$

dual basis

- given our basis for vectors, we have a natural dual basis $(du^i)_a$, with

$$(du^i)_a (\frac{\partial}{\partial u^j})^a = \delta^i_j$$

- lets take the w_i numbers from above and write a covector using coordinates

$$x_a = \sum_i w_i (du^i)_a$$

- we can compute in coordinates:

$$x_a v^a = \sum_i w_i v^i$$

- so $x_a = w_a$
- so $w_a v^a = \sum_i w_i v^i$, where w_i and v^i are the coordinates in a fixed basis.
  - so we now know how to caculate the coordinates of a covector in the dual basis. Simply apply the co-vector to the primal basis vectors.

dual change of paremeterization

- given our earlier change of parametrization (same J matrix), we have the the associated dual transformation rule

$$(du^i)_a = \sum_j J^i_{\hat{j}} (d\hat{u}^{\hat{j}})_a$$

- so the same J can also be used to transform covector coordinates wrt "u" to coordinates wrt $\hat{u}$.

$$\begin{aligned} w_a &= \sum_i w_i (du^i)_a \\ &= \sum_i w_i \sum_j J^i_{\hat{j}} (d\hat{u}^{\hat{j}})_a \\ &= \sum_j (d\hat{u}^{\hat{j}})_a \sum_i w_i J^i_{\hat{j}} \\ &= \sum_j (d\hat{u}^{\hat{j}})_a w_{\hat{j}} \end{aligned}$$

- and so

$$w_{\hat{j}} = \sum_i w_i J^i_{\hat{j}}$$

  - note the directions.
- to complete the dictionary, we would have to compute the inverse matrix to J.

tensors 1

- a covector w_a takes one vector and gives us a number linearly
- we can define scalar multiply and addition for covectors.
- conversely a vector v^a takes one covector and gives us a number linearly
- we can generalize this.
- a tensor w_{ab} takes in an ordered pair of two vectors and give out a number, bilinearly.
  - here we are naming the first slot a and the second slot b.
  - superscripts on the input vectors tell us which slot to put them in.
- let w^{ab} take in two covectors and give out a number, bilinearly.
- let w^b_a take in one vector and one covector and give out a number
  - takes a vector and covector and gives a number
  - takes a vector and gives out a vector
  - such an object can have eigenvectors
- one (not the only) way to get these is by taking an outerproduct say $m_{ab} := w_a v_b$

tensors 2

- this basic idea can be generized to (i, j) tensors that take vectors and co-vectors and turn them in to numbers. eg T^{abc}_{de}
- once we pick a basis (which gives us a dual basis as well), we can represent a tensor with coordinates T^{ijk}_{lm}.
  - a "matrix" with one dimension for each slot.
- in order to change to the hat basis, we sum against $J^m_{\hat{m}}$ on the subscripts and its inverse $K^{\hat{i}}_i$ against the superscripts.

computing

- to apply to input co/vectors, as $T^{ab}_d v^d w_a x_b$, we can compute this as $\sum_{ijk} T^{ij}_k v^k w_i x_j$
  - answer is basis invariant
- we can also define $N^a_b M^b_c$ as the $(1, 1)$ tensor represented by $\sum_j N^i_j M^j_k$
  - answer is basis invariant
- we can also define contraction of an upper against a lower N^a_a using $\sum_i N^i_i$.

- also basis invariant!

first fundamental form

- suppose i want to take the dot product in R^3 between two tangent vectors, v^a, and w^b.
- the dot operator takes two tangent vectors and gives out a number.
- the dot operator is "linear" in each of its two slots.
- so we can write such an operator as g_{ab}
  - g is the preferred letter for this operator.
  - defined at each point in the surface.
- we write the dotting as

$$g_{ab}\; v^a w^b$$

- the a,b tell us which slot of "g" to put in "v" and "w", so the order of how we list the symbols is not relevant
- it also happens to be the case that g is symmetric.

$$g_{ab} = g_{ba}$$

  - question can a tensor T^a_b be symmetric?
- g is called the metric tensor of the surface.
- we define the First fundamental form, which maps a tangent vector to its squared length

$$I(v^a) := g_{ab} v^a v^b$$

metric in coordinates

- suppose we write our two vectors written with coordinates in a basis

$$v^a = \sum_i (\frac{\partial}{\partial u^i})^a v^i$$
$$w^b = \sum_j (\frac{\partial}{\partial u^j})^b w^j$$

- then we can compute the dot of any two vectors as

$$\begin{aligned} g_{ab}\; v^a w^b &= g_{ab}(\sum_i (\frac{\partial}{\partial u^i})^a v^i)(\sum_j (\frac{\partial}{\partial u^j})^b w^j) \\ &= \sum_i \sum_j v^i w^j g_{ab} (\frac{\partial}{\partial u^i})^a (\frac{\partial}{\partial u^j})^b \\ &= \sum_i \sum_j g_{ij} v^i w^j \end{aligned}$$

  where

$$g_{ij} := g_{ab}\; (\frac{\partial}{\partial u^i})^a (\frac{\partial}{\partial u^j})^b$$

- remember, these 4 numbers g_{ij} depend on our choice of parametrization/basis
- if we had chosen a different basis, then we would have a different set of numbers $\hat{v}^i$, $\hat{w}^j$ $\hat{g}_{ij}$ that represent the same tensors.

isothermal coordinates

- not amazing fact: one can always find a parameteriztion where $g_{11} = g_{22} = 1$, and $g_{12} = 0 = g_{21}$. at a single point.

- this is a useful tool

- isothermal parametrization: where $g_{11} = g_{22} = \lambda^2$, and $g_{12} = 0 = g_{21}$.
- tangent basis is ortho with same scale
  - λ is spatially varying
- amazing fact: one can always find an isothermal parametrization over a patch

area of patch

- fix our parametrization u^i
- compute our 4 numbers g_{ij} at every point
- put them into a matrix G at every point
- we can compute area as

$$A = \int_\Omega \int \sqrt{det(G)} \; du^1 du^2$$

inverse metric

- there is a well defined identity tensor I^a_b.
  - not true of other script types.
- we can define an inverse metric g^{ab} to be the tensor with the property $g^{ab}g_{bc} = I^a_c$.
  - we use the same letter g, but now it has superscripts.

curves on surfaces

- we can represent a curve on a surface as

$$c(t) = x(u^1(t), u^2(t))$$

- with two functions $u^i(t)$
- we can compute its tangent vector

$$(\frac{dc}{dt})^a = \sum_i (\frac{\partial x}{\partial u^i})^a \frac{du^i}{dt}$$

- so we see that the coordinates of the tangent vector $(\frac{dc}{dt})^a$ in the basis $(\frac{\partial}{\partial u^i})^a$ are

$$\frac{du^i}{dt}$$

directional derivative

- suppose one has a real valued function f over every point on the surface.
- here is how to define the directional derivative of f as one moves in the tangent direction v^a.
- we find some curve $c(t) = x(u^1(t), u^2(t))$, that has its tangent v^a at the point in question
- this defines a scalar function $f(t)$ which we can differentiate

$$\frac{df}{dt} = \sum_i \frac{\partial f}{\partial u^i}\frac{du^i}{dt}$$
$$= \sum_i v^i \frac{\partial f}{\partial u^i}$$

- note that the result only depends on the tangent of the curve!
- at a point, and fixing f, the result is a linear functional of the tangent direction.
- so we write the directional derivative as

$$v^a(\partial_a f)$$

- where $\partial_a f$ is this linear functional and so has meaning even before we contract it with v^a.

kinds of derivatives

- we get the same answer no matter how we parametrize, so we use the generic symbol
  - since we were just computing ds/dt.

$$v^a(\nabla_a f)$$

- it also can be written as $v^a(df)_a$, $D_v f$, or $v^a[f]$
- ∂_a means partial derivative operator. ∇_a means covariant derivative. little d here means exterior derivative. D_v means directional derivative. we can even just think of this as simply "applying" the vector field to f.
  - in this interpretation, we can think of tangent vectors, not as geometry, but as a operator on functions that behaves like a derivative. that is why we used the basis notation.
- when applied to scalar functions over a surface, they all do the same thing.

gradiant

- the gradiant of f is a vector w^a such that $g_{ab}w^a v^b = v^a \nabla_a f$
- clearly the gradiant is $g^{ab}\nabla_b f$
- note: gradiant vector needs the metric for its definition, unlike the derivative co-vector.
- (only) in an orthonormal basis, $w^i = \frac{\partial f}{\partial u^i}$

steepest ascent

- gradiant vector gives the direction of steepest ascent

$$\max_{x | x^a g_{ab} x^b = 1} x^a \nabla_a f$$

- work in orthonormal basis

$$\max_{x | \sum_i x^i x^i = 1} \sum_i x^i \frac{\partial f}{\partial u^i}$$

- maximized when the coordinate-vector x^i is a scale of the coordinate-vector $\frac{\partial f}{\partial u^i}$
- which, in these coordinates, is the gradiant direction
- notice: that the direction of steepest ascent requires the use of the metric.

derivative of normal

- let call the unit normal n.
  - think of it as 3 scalar functions.
- lets look at all the directional derivatives of n on the surface.
- this gives us a mapping from from the tangent plane to a vector in R^3

- fact: the derivative of normal as we move along any tangent direction always lies in the (embedded) tangent plane of the surface point.
  - intuition: change of a unit length vector field is orthogonal to the vector itself

shape operator

- so we now have a mapping from tangent vectors to tangent vectors.
- the mapping is linear in the input.
- we negate this and call this mapping the shape operator S^c_a
- the scripts means that it takes in one tangent vector and outputs one tangent vector.
- so the negated normal derivative as one moves along a tangent vector v^a is

$$z^c = S^c_a v^a$$

second fundamental form

- lets define an operater that takes the the negated normal derivative in the direction of one vector v^a, and then dots it with a second vector w^b to obtain a number

$$w^b g_{cb} S^c_a v^a$$

- this defines a billinear form on a pair of vectors, which we can represent as

$$q_{ab} := g_{cb} S^c_a$$

- fact: q_{ab} happens to be symmetric.
- in this case we say that S^c_a is self adjoint (wrt the metric).
- we define the second fundamental form operating on a single tangent vector as

$$II(v^a) := q_{ab} v^a v^b$$

- amazing fact (bonnet): if two surfaces have the same I and II at all points, then they must be the same geometry up to a euclidean transform.

normal sections and curvature

- define the normal curvature in the direction v^a.
- as $k_n(v^a) := II(v^a)/I(v^a)$

$$\frac{g_{bc} S^b_a v^a v^c}{g_{ac} v^b v^c}$$

- suppose i intersect a surface with a plane that includes the normal and one tangent vector v^a at a point
- interpretation, the normal/tangent sliced with the surface gives us a planar curve.
- this curve has a well defined notion of curve-curvature (uses osculating circle).
- fact: this curve's curvature agrees with the just defined normal curvature!

principle curvatures

- i'd like to get a handle on the normal curvature as i spin the tangent vectors around.
- compute eigenvalues k_1, k_2 and eigenvectors e^a_1, e^a_2 of S^a_b
  - parametrization independent
  - they exist since S^a_b is self adjoint

- and evecs will be orthogonal wrt g_{ab} (spectral theorem)

- call these principal curvatures and principal directions
- no normal twisting as i walk in these two directions!
- min-max thm; : the extrema of $II(v^a)$ subject to $I(v^a) = 1$ are in the principal directions

details: spectral thm

- work in orthonormal coordinates,
  - g_{ab} represented by the identity.
- (a) q_{ab} as well as (b) S^a_b are now represented by a matrix S.
  - (a) tells us that S is a symmetric matrix
  - (b) tells us that the eigevectors of S give us the coordinates of the eigenvectors of S^a_b.
- from matrix theory, since S is symmetric, it has a real eigendecomposition, $S = U^t DU$, where the columns of U^t are orthogonal in E^2.
- these are the coordinates of the eigenvectors of S^a_b and are clearly orthogonal wrt g_{ab}.
  - this gives us the spectral theorem

details: min max thm

- in these coordiates, our optimization problem is $\max_x \; x^t Sx$ s.t. $x^t x = 1$

$$\max_x \frac{x^t Sx}{x^t x}$$

$$\max_x \frac{(x^t U^t)D(Ux)}{(x^t U^t)(Ux)}$$

$$\max_u \frac{(u^t)D(u)}{(u^t)(u)}$$

$$\max_u \frac{\sum_i \lambda_i u_i^2}{\sum_i u_i^2}$$

- now

$$\frac{\sum_i \lambda_i u_i^2}{\sum_i u_i^2} \leq \frac{\sum_i \lambda_M u_i^2}{\sum_i u_i^2} = \lambda_M \frac{\sum_i u_i^2}{\sum_i u_i^2} = \lambda_M$$

- max is achieved if x is max eigenvector
- similar for min

gaussian curvature

- we call the product $k = k_1 k_2$ the gaussian curvature at a point
- its sign -0+ determines if we call the point hyperbolic (saddle), parabolic, or elliptical (convex or concave)
- so the surface is broken up into hyperbolic, convex, and concave regions, separated by parabolic curves.
- amazing fact (gauss): gaussian curvature can be computed completely from knowing g_{ab} and its derivatives.
- amazing fact (gauss-bonnet): if i have a closed surface, then the area integral of its gaussian curvature is completely determined by its euler characteristic.

mean curvature

- $H = 1/2(k_1 + k_2)$ is called mean curvature

- a patch is called a minimal surface if it has zero mean curvature.
  - must be saddle everywhere.
  - fact: can't make its area smaller by deforming it
- if one has an isothermal paramterization, then the normal vector, with length of the mean curvature can be computed using the laplacian

$$\triangle x := \sum_i \frac{\partial^2 x}{\partial u^i \partial u^i} = 2\lambda^2 Hn$$

- there is actually an operator called laplace-beltrami, which can be defined without fixing any special parametrization, which also gives Hn

conjugacy

- we say two tangent vectors v^a, w^b are conjugate if

$$q_{ab}v^a w^b = 0$$

- it means that the as i move in direction v, the normal turns in a direction that is orthogonal to w.

conjugacy and silhouettes

- a silhouette point on a surface has its normal orthogonal to the viewing direction.
- this means that the viewing vector is a tangent vector at a silhouette point, so we can call it v^a.
- the silhouette curve on the surface separates out regions that face towards and away from the viewer.
- so the silhouette must generally form a smooth simple closed curve on the surface.
- the silhouette curve has a tangent direction, say w^b.
- as we move in the direction w^b, the normal stays orthogonal to the view ray, so it can't be bending towards or away from v^a

$$q_{ab}v^a w^b = 0$$

- so these must be conjugate directions.
- conjugacy also shows up when we look at isophotes of shading as compared to the direction to the light.

asymptotics and cusps of silhouettes

- if a tangent is self conjugate, we call it an asymptotic direction

$$q_{ab}v^a v^b = 0$$

- only "hyperbolic" regions have asymptotes
- as i walk in an asymptotic direction, the normal "twists"
- for a smooth silhouette to look non-smooth in projection, it must be the case that we are looking at the silhouette curve "edge on"
- the silhouette tangent equals the view direction.
- the view ray must be an asymptotic direction.
- because of visibility, this is where the silhouette ends in the image

gauss map

- the space of normal directions can be thought of as points on a sphere
- using the normal, we can map each point to the sphere, called the gauss map $n(p)$.

- elliptical regions map to the sphere with preserved oritentation and hyperbolic regions map to the sphere with flipped orientation
- so the gauss map has folds along parabolic curves.
- sillhouettes (orthogonal view) are the preimage of a great circle on the gauss sphere
- so "lips and beaks" silhouette singularities must happen at parabolic points.

derivatives of mappings

- suppose i have a mapping m from one surface $\hat{S}$ to another S.
- then if i place a curve on the first surface, it maps to a curve on the second surface.
- so i can look at how the tangent vector of one curve at a point maps to a tangent vector at $m(p)$
- fact: it does not make a difference how i chose the curve, as long as it has the desired tangent on the first surface. so we have a differential mapping between the two tangent planes
- fact: the differential mapping is linear in the input tangent vectors
- so we can write this mapping as $dm^b_{\hat{a}}$
- the hats tell us that it maps a hatted vector to an unhatted vector.

gauss map

- the differential of $n(p)$ can be written as $dn^b_{\hat{a}}$
- the tangent of the gauss sphere must match the tangent at a point.
- so we can identify both tangent planes making $dn^{\hat{b}}_{\hat{a}}$ a linear mapping from a tangent plane to itself.
- this is another way to define the shape operator $S^{\hat{b}}_{\hat{a}} = -dn^{\hat{b}}_{\hat{a}}$