context

- given a mapping m from $(\hat{S}, g_{\hat{a}\hat{b}})$ to (S, g_{ab})
- for paramterization of a disk like surface S, $\hat{S}$ will be a disk-like portion of the plane.
- in practice S will be a triangle mesh
- we will map each mesh triangle to one triangle in the plane ' (One Piece linear) 'O.P.L."
- we will think of the postions of the vertices in the plane as the eventual unknowns.

pullback

- we have the differential tangent map $dm^b_{\hat{a}}$
- we define the pull back of g as

$$g_{\hat{a}\hat{b}} \equiv dm^d_{\hat{b}}\, g_{cd}\, dm^c_{\hat{a}}$$

- this allows us effectively apply g to tangent vectors of $\hat{S}$.

isometry

- we call a mapping m from $(\hat{S}, \hat{g}_{\hat{a}\hat{b}})$ to (S, g_{ab}) isometric if

$$\hat{g}_{\hat{a}\hat{b}} = g_{\hat{a}\hat{b}}$$

- this means that the dot product is presereved under the mapping.
- so are lenghts, and so are all "distances" between points
- most pairs of surfaces are do not have an isometric mapping between them

comformal

- we call a mapping m from $(\hat{S}, g_{\hat{a}\hat{b}})$ to (S, g_{ab}) conformal if

$$\lambda^2 \hat{g}_{\hat{a}\hat{b}} = g_{\hat{a}\hat{b}}$$

- for some spatially varying λ
- this means that at a point, all dot products are just scaled
- this implies that angles are preserved
- this constraint places $3 - 1 = 2$ constraints on the mapping at each point
  - symmetric 2 by 2 matrix has 3dof, scale has 1
- deep fact: there is a conformal map between any two disc-like surfaces
  - unique up to 3 dofs.
  - so you cannot fix the mapping at the boundary.
  - if we require the map to be O.P.L. we will most certainly fail

benefits of conformal

- preserves "shape", but areas are not controlled
- if sampling rate can be adaptive but istropic (quad tree), then can overcome the area distortion.
- preserves aspect ratios of triangles
- lots of continuous theory

area preserving

- we call the map area preserving if

$$dA_g = dA_{\hat{g}}$$

- this imposes one constraint on the mapping at each point
- fact: there are loads of area preserving maps between pairs of disks.
- e.g. you can fix the boundary mapping
- you can add other criterea to optimize as well
- if you require the map to be O.P.L. you may or may not fail.
- not pursued that much in the literature
- this is also useful for cartograms.

stretch

- define the (squared) stretch of a vector $v^{\hat{a}}$ as

$$\frac{v^{\hat{a}}\ g_{\hat{a}\hat{b}}\ v^{\hat{b}}}{v^{\hat{a}}\ \hat{g}_{\hat{a}\hat{b}}\ v^{\hat{b}}}$$

- it is the size of the unitized vector under m
- we wish to understand the stretch for various directions
- we want to figure out the directions of max and min stretch
- this will lead to a notion of non-conformal energy to minimize

dirchilet tensor

- define the "dirichilet" tensor as

$$\begin{aligned} p^{\hat{a}}_{\hat{e}} &\equiv \hat{g}^{\hat{a}\hat{b}}\ dm^c_{\hat{b}}\ g_{cd}\ dm^d_{\hat{e}} \\ &= \hat{g}^{\hat{a}\hat{b}}\ g_{\hat{b}\hat{e}} \end{aligned}$$

- $p^{\hat{a}}_{\hat{e}}$ is self adjoint wrt $\hat{g}_{\hat{a}\hat{b}}$ since $p^{\hat{a}}_{\hat{e}}\hat{g}_{\hat{a}\hat{b}} = g_{\hat{b}\hat{e}}$ is symmetric
- so by the spectral thm, p has 2 eigenvectors that are ortho wrt $\hat{g}$.
- these are the max and min of

$$\frac{v^{\hat{a}}\ g_{\hat{a}\hat{b}}\ v^{\hat{b}}}{v^{\hat{a}}\ \hat{g}_{\hat{a}\hat{b}}\ v^{\hat{b}}} = \frac{\hat{g}_{\hat{a}\hat{c}}\ p^{\hat{c}}_{\hat{b}}\ v^{\hat{a}}\ v^{\hat{b}}}{\hat{g}_{\hat{a}\hat{b}}\ v^{\hat{a}}\ v^{\hat{b}}}$$

properties of p

- call the two orthogonal (wrt $\hat{g}$) eigenvectors $n^{\hat{e}}_i$ with positive eigenvalues $\gamma_1^2 \geq \gamma_2^2$
- stretch is maximzied for vectors in direction $n^{\hat{e}}_1$ and minimized for $n^{\hat{e}}_2$
- easy fact: we have $\frac{dA_g}{dA_{\hat{g}}} = \gamma_1\gamma_2 = \sqrt{DET(P)}$
  - where P is the matrix expression of p using any set of coordinates

non-conformality energy

- for a conformal map, we must have $\gamma_1 = \gamma_2$

- so one way to measure non-conformality is

$$\int_{\hat{S}} dA_{\hat{g}}\ (\gamma_1 - \gamma_2)^2 =$$
$$\int_{\hat{S}} dA_{\hat{g}}\ (\gamma_1^2 + \gamma_2^2) - 2\int_{\hat{S}} dA_{\hat{g}}\ (\gamma_1\gamma_2) =$$
$$\int_{\hat{S}} dA_{\hat{g}}\ (\gamma_1^2 + \gamma_2^2) - 2\int_{\hat{S}} dA_g =$$
$$\int_{\hat{S}} dA_{\hat{g}}\ (\gamma_1^2 + \gamma_2^2) - 2A(S)$$

- if we are optimizing this energy over maps where S is fixed, then so is its area, and so we may as well just optimize the "dirichilet" energy:

$$P = \int_{\hat{S}} dA_{\hat{g}}\ (\gamma_1^2 + \gamma_2^2)$$

- dirichilet energy $\equiv$ non-conformality + 2 area of range

variational

- we can look at some space of mappings and minimize the dirichilet energy over this space
- if we lock the down the mapping along the boundary, we cannot get a conformal map.
- the minimizer is called "harmonic"

dirichilet energy

- we can compute the γ_i by picking any coordinates, representing p as a matrix, and diagonalizing $P = E^{-1}DE$
- the quantity $(\gamma_1^2 + \gamma_2^2)$ is trace(D).
- but trace(D)=trace(P)= $p_{\hat{a}}^{\hat{a}}$.
- so we can also write the dirchilet energy as

$$\int_{\hat{S}} dA_{\hat{g}}\ p_{\hat{a}}^{\hat{a}} = \int_{\hat{S}} dA_{\hat{g}}\ \hat{g}^{\hat{a}\hat{b}}\ dm_{\hat{b}}^{c}\ g_{cd}\ dm_{\hat{a}}^{d}$$

conformal invariant

- we are looking at the dirichilet energy of a mapping from $\hat{S}$ to S
- if I also have a conformal mapping from $\hat{\hat{S}}$ to $\hat{S}$.
- this gives me a composed map from $\hat{\hat{S}}$ to S.
- fact: the energy of the composed map is the same as the energy of the original map
- related to the following: swap the metric $\hat{g} = \lambda^2 \hat{g}_{\hat{a}\hat{b}}$
- so dirichilet integrand (which has a factor of $\hat{g}^{\hat{a}\hat{b}}$) scales with $1/\lambda^2$, but $dA_{\hat{g}_{\hat{a}\hat{b}}}$ scales with λ^2
- relevance: if $(\hat{S})$ is allowed to scale, the optimization will not im/explode it.

MIPS (corrected?)

- if the mapping is P.L., then dm is constant per triangle so dirichilet energy of the mapping is

$$P = \int_{\hat{S}} dA_{\hat{g}}\ (\gamma_1^2 + \gamma_2^2)$$
$$= \sum_{t \in T} A(\hat{t})(\gamma_1^2(t) + \gamma_2^2(t))$$

- where $A(\hat{t})$ is the area of the triangle in the plane.
- this will end up being some non-quadradic expression in the (u_i) positions of the vertices.
- non-linear numerical optimization
- for continuous case, any conformal map to any region of the plane will have Dirichilet energy $P = 2A(S)$.
- for OPL, we will not have any confomral maps, so we will have some non-confomral energy.
  - so some boundary shapes in the plane will lead to less energy than others
- but due to conformal invariance, it will not try to implode so one can just leave the boundaries free
- cost goes infinite at degenerate area triangle, so will avoid flips (i think).
- i call this MIPS
  - MIPS as oringinally published is a bit different, and did not take into account the triangles areas, and so is not intrinsic.

Eck

- we also have an inverse mapping $u := m^{-1}$ from S to $\hat{S}$
- one can compute the dirichilet energy of that inverse mapping
- the extrema of the stretch of the inverse mapping are $\lambda_i = 1/\gamma_i$
- we will still have $\hat{S}$ be the plane.
- so the dirichilet energy of the inverse mapping can be computed as

$$\int_S dA_g \; (\lambda_1^2 + \lambda_2^2) =$$
$$\sum_{t \in T} A(t)(\lambda_1^2 + \lambda_2^2)$$

- where $A(t)$ is 3d area of the triangle.

Eck properties

- -: dirichilet = non-conformal + area of range
  - so dirichilet of inverse mapping will want to implode the paramter domain
- so fix the boundary
- could allow to slide along boundary, but this would be complicated to constrain
- so just fix the boundary mapping
- we won't get a conformal map, but the optimal is called a "harmonic map".
- in regions that are very non-conformal, Eck will try to shrink, so there are some quality issues
- this is called Eck.

Eck as poisson problem

- $\hat{S}$ is the plane, so lets use planar coordinates u_1, u_2.
- then the energy is

$$\int_S dA_g \; g^{ab} \; du_b^{\hat{c}} \; \hat{g}_{\hat{c}\hat{d}} \; du_a^{\hat{d}} = \sum_{\hat{i}} \int_S dA_g \; g^{ab} \; du_b^{\hat{i}} \; du_a^{\hat{i}}$$

- this is exactly the same energy from the sum of 2 poisson problems, with a zero right hand side!!

PL case

- since S is a mesh, we can write this as

$$\begin{aligned}\sum_i ((u^i)^t d^t M_2 du^i) &= \sum_i 1/2 \sum_t \sum_{he_{vw}} (\cot \alpha_{vw})(u^i_v - u^i_w)^2 \\ &= \sum_i 1/2 \sum_{e_{vw}} (\cot \alpha_{wv} + \cot \alpha_{vw})(u^i_v - u^i_w)^2\end{aligned}$$

- +: the energy is quadradic in the vertex position
- can be written as spring energy with symmetric "cotangent weights"
  - lots of theory for this
  - can be negative weights (so possible flips).

levy-Desbrun

- go back to the original non-conformal energy of the inverse mapping

$$\int_S dA_g \ (\lambda_1 - \lambda_2)^2$$

- this likes conformality. if not conformal, will try to shrink, but not at the expense of conformality.
- in smooth case, any conformal map will have zero energy
- for O.P.L. there is no conformal map, and so minimum solution is a full collapse.
  - unless it was flat to begin with
- must add at least 2 fixed vertex constraints. (choice changes the result).
- this will not want to collapse
- this is still a linear system (see below)!
  - so is strictly better than Dirichilet.
  - for O.P.L., we still get cotangent springs at interior vertiecs and linear but non seperable equations at the boundary vertices
- linear system, may have flips due to negative weights, or due to the non-convexity of the boundary.

as similar as possible

- now, to obtain another interpretation for levy....
- lets set up a new problem
- for each mapping u, and spatially varying similarity transform C, define an energy as

$$\int_S dA_g \ (du^{\hat{a}}_b \ - C^{\hat{a}}_b)(du^{\hat{c}}_d \ - C^{\hat{c}}_d) g^{bd} \hat{g}_{\hat{a}\hat{c}}$$

- since $\hat{S}$ is the plane, we get

$$\sum_{\hat{k}} \int_S dA_g \ (du^{\hat{k}}_b \ - C^{\hat{k}}_b)(du^{\hat{k}}_d \ - C^{\hat{k}}_d) g^{bd}$$

- minimize over u and C.
- if u is conformal mapping, there will be a C that gives us zero energy
- this is a sum of two poisson energies, where the C^k is the right hand side.

rewrite

- so for ASAP, on a triangle mesh, we get

$$\sum_{\hat{k}} 1/2 \sum_t \sum_{he_{vw}} (\cot \alpha_{vw})((u_v^{\hat{k}} - u_w^{\hat{k}}) - \sum_l C_l^{\hat{k}}(t)[x_v^l - x_w^l])^2$$
$$= 1/2 \sum_t \sum_{he_{vw}} (\cot \alpha_{vw})||((u_v - u_w) - C(t)[x_v - x_w])||^2$$

- in the last expression, $C(t)$ is a 2-by-2 matrix

PL ASAP

- for P.L. mapping, and one C per triangle
  - parameterize as $[[a, b], [-b, a]]$
- we get a linear system in our variables!

interpretation of ASAP

- C is just a helper variable. we want to understand this as an optimization over the mapping u.
- define $E(u) = \min_C E(u, C)$
- then $\min_{u,C} E(u, C) = \min_u E(u)$
- if we understand $E(u)$, then we understand ASAP.
- define $C^*(u) = \text{argmin}_C E(u, C)$,
- then $E(u) = E(u, C^*(u))$
- so if we understand $C^*(u)$, we understand ASAP.

procrustus

$$\min_C \int_S dA_g \; (du_b^{\hat{a}} \; - C_b^{\hat{a}})(du_d^{\hat{c}} \; - C_d^{\hat{c}}) g^{bd} \hat{g}_{\hat{a}\hat{c}}$$

- fixed du, and C must be a similarity transform
- can just look at each point (or triangle) at a time
- and fix parameters at this point so that both sides are orthonormal and get a matrix problem

$$\min_C |du_j^{\hat{k}} \; - C_j^{\hat{k}}|_F^2$$

- note that in these coords, $P^{-1} = du^t * du$.
- by (signed) SVD:

$$du_j^{\hat{k}} = U \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} V^t$$

- one λ is negative if du is a flip
- by Procrustus:

$$C^* = \frac{1}{2} U \begin{bmatrix} \lambda_1 + \lambda_2 & 0 \\ 0 & \lambda_1 + \lambda_2 \end{bmatrix} V^t$$

- if you plug this C^* you get

$$E(u) = \int_S dA_g \; (\lambda_1 - \lambda_2)^2$$

  - use the permuted trace rule (see paper)

- thus we are back to Levy-Desbrun.

ARAP

- if we replace the similarity C, with a rotation R, we will get "as rigid as possible" parameterization
- by Procrustus:

$$R^* = UV^t$$

- plug this in

$$\int_S dA_g\ (\lambda_1 - 1)^2 + (\lambda_2 - 1))^2$$

ARAP computation
R is subject to non linear constraints, so this will be a non linear problem
in PL case, if we fix R, we get a poisson system to find best u. and for fixed u can find best R per triangle using procrustus so can iterate back and forth.
green energy

- measures internal stresses
- tries to find an isometry
- generally it cannot succeed

$$\begin{aligned} &\int_S dA_g\ |\hat{g}_{ab} - g_{ab}|^2 \\ = &\int_S dA_g\ (\hat{g}_{ab} - g_{ab})(\hat{g}_{cd} - g_{cd})g^{ac}g^{bd} \\ = &\int_S dA_g\ (1 - \lambda_1^2)^2 + (1 - \lambda_2^2)^2 \end{aligned}$$

- non linear
- doesn't mind flips