assignment: Lorenz chapters 1 and 3

1. Generate a random N x M matrix, say with N=20 and M=10, and call it $\mathbf{W}_{ij}$. This matrix represents the spatial and temporal structure of the quantity $W(x,t)$ collocated on points in space (i component) and time (j component). You may consider points in space to be separated by $\delta x$ and points in time by $\delta t$. Use the red noise process $W_{i+1, j} = \alpha W_{ij} + \epsilon_j$ with $\epsilon_j$ a unit variance zero mean random number with a Gaussian distribution to produce temporal correlation and $W_{i+1, j} = \beta W_{ij} + \epsilon_i$ to produce spatial correlation. Try $\alpha = .5$ and $\beta = .9$.

a) Form the temporal mean at each space point $\overline{W}_i = \frac{1}{M} \sum_{j=1}^{M} W_{ij}$; the spacial mean at each time point $[W]_j = \frac{1}{N} \sum_{i=1}^{N} W_{ij}$; the departure from the temporal mean at every space and time point $W_{ij}' = W_{ij} - \overline{W}_i$; and the departure from the spacial mean at every space and time point $W_{ij}^* = W_{ij} - [W]_j$.

b) Show that $\mathbf{W}_{ij} = [\mathbf{W}] + \mathbf{W}_i^T + [\mathbf{W}]_j^T + \mathbf{W}_{ij}'$. That is, the total field in space and time can be decomposed into a scalar space and time mean plus a vector time average of departures from the spatial mean plus a vector space average of the departure from the the time mean plus a matrix departure at each space and time point from the space and time mean. Show this in general and also verify it for your example.

c) Find the temporal mean of the square departures from the temporal mean of $\mathbf{W}$; that is $\mathbf{V}_i = (\mathbf{W}_{ij}')^2$; this is the variance of $\mathbf{W}$. Form another N x M matrix $\mathbf{Q}_{ij} = \mathbf{W}_{ij} + \epsilon_{ij}$ where $\epsilon_{ij}$ is a matrix of Gaussian random numbers with zero mean and unit variance. Find the temporal mean covariance of the departures of $\mathbf{W}_i$ and $\mathbf{Q}_i$ from their respective temporal means, $\mathbf{C}_i = \mathbf{W}_{ij}' \mathbf{Q}_{ij}'$. Show that $\overline{\mathbf{W}}_i \mathbf{Q}_{ij} = \overline{\mathbf{W}}_i \overline{\mathbf{Q}}_i + \mathbf{C}_i$ and interpret this relation physically for an example in which $\mathbf{W}$ is the vertical component of velocity wind and $\mathbf{Q}$ is the water vapor content of the air.

d) Form the time mean spatial covariance matrix $\mathbf{R}_{ik} = \mathbf{W}_{ij} \mathbf{W}_{kj} - \overline{\mathbf{W}}_i \overline{\mathbf{W}}_k$; obtain the eigenvalues and eigenvectors of $\mathbf{R}$ and interpret these physically.

Hint for part d:

The covariance matrix $\mathbf{R}_{ik}$ contains the temporal mean of the spatial covariance of the variable $\mathbf{W}_i$. This matrix contains all information on the temporal mean second order spatial statistics of $\mathbf{W}$. Its eigenvectors provide the most parsimonious basis for representing $\mathbf{W}$. By the Eckart-Young-Mirsky theorem the most parsimonious representation at order $K < N$ of $\mathbf{R}$ is truncation at order $K$ of its singular value decomposition:

$$\mathbf{R}_{ik} = \sum_{l=1}^{N} \mathbf{w}_i \lambda_l \mathbf{w}_k$$

in which the vectors $\mathbf{w}_1$ (called empirical orthogonal functions or EOF’s) are orthogonal and the $\lambda_l$ are conventionally placed in decreasing order. The $\mathbf{w}_1$ are the eigenvectors of $\mathbf{R}$ and the $\lambda_l$ are the corresponding eigenvalues. Because of their orthogonality, the structure $\mathbf{w}_1$ accounts individually for $\lambda_l$ of the total variance and their sum accounts for all of the variance. Truncation leaves out the variance accounted for by the sum of the truncated eigenvalues $\lambda_l$ and the spatial structures of their associated eigenvectors $\mathbf{w}_l$. By extension $\mathbf{W}_{ij}$ itself is most parsimoniously represented by truncation of: $\mathbf{W}_{ij} = \sum_{l=1}^{N} \alpha_{lj} \mathbf{w}_{il}$, in which the temporal variation of the projection of the variable on the EOF’s $\mathbf{w}_{il}$ are called the Principal Components $\alpha_{lj} = \sum_{i=1}^{N} \mathbf{w}_{il} \mathbf{W}_{ij}$.

The utility of this truncation is that typical geophysical datasets have a very high spatial dimension but only a small subspace of this space is varying appreciably in time; EOF analysis identifies this subspace.