Name:

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| MWF 9 Jameel Al-Aidroos |
| MWF 9 Dennis Tseng |
| MWF 10 Yu-Wei Fan |
| MWF 10 Koji Shimizu |
| MWF 11 Oliver Knill |
| MWF 11 Chenglong Yu |
| MWF 12 Stepan Paul |
| TTH 10 Matt Demers |
| TTH 10 Jun-Hou Fung |
| TTH 10 Peter Smillie |
| TTH 11:30 Aukosh Jagannath |
| TTH 11:30 Sebastian Vasey |

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3,8, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

| | | |
|---|---|---|
| 1 | | 20 |
| 2 | | 10 |
| 3 | | 10 |
| 4 | | 10 |
| 5 | | 10 |
| 6 | | 10 |
| 7 | | 10 |
| 8 | | 10 |
| 9 | | 10 |
| 10 | | 10 |
| Total: | | 110 |

Problem 1) True/False questions (20 points), no justifications needed

1) T F There is a function $f(x,y)$ for which the linearization at $(0,0)$ is $L(x,y) = x^2 + y^2$.

2) T F For any two functions $f, g$ and unit vector $\vec{u}$ we have $D_{\vec{u}}(f+g) = D_{\vec{u}}f + D_{\vec{u}}g$.

3) T F $\int_0^2 \int_0^{\sqrt{4-x^2}} (x^2+y^2)\, dydx = \int_0^2 \int_0^{\pi/2} r^2\, d\theta dr$.

4) T F If we solve $\sin(y) - xy^2 = 0$ for $y$, then $y' = -y^2/(\cos(y) - 2xy)$.

5) T F If $f(x,0) = 0$ for all $x$ and $f(0,y) = 0$ for all $y$, then $g(x,y) = \int_0^x \int_0^y f(s,t)\, dtds$ solves $g_{xy}(x,y) = f(x,y)$.

6) T F If $|\nabla f| = 1$ at $(0,0)$, then there exists a direction in which the slope of the graph of $f$ at $(0,0)$ is 1.

7) T F The function $f(x,y) = x^2 + y^2$ satisfies the partial differential equation $f_{xx}f_{yy} - f_{xy}^2 = 4$.

8) T F The height of Mount Wachusett is $f(x,y) = 4 - 2x^2 - y^2$. On the trail $x^2 + y^2 = 1$, the point $(1,0)$ is a maximum.

9) T F Mount Wachusett has height $f(x,y) = 4 - 2x^2 - y^2$. Except at the maximum $(0,0)$, the gradient vector is perpendicular to the graph of the function.

10) T F If $f_x(a,b) > 0$ and $f_y(a,b) > 0$ then for any unit vector $\vec{u}$ we must have $D_{\vec{u}}f(a,b) > 0$.

11) T F If $f(x,y)$ has two local minima, then $f$ must have at least one local maximum.

12) T F If $\vec{r}(t)$ is a curve on the surface $g(x,y,z) = x^2 + y^2 - z^2 = 6$ then $\nabla g(\vec{r}(t)) \cdot \vec{r}\,'(t) = 0$.

13) T F If $f$ and $g$ have the same trace $\{x = 5\}$ then $f_x(5,y) = g_x(5,y)$ for all $y$.

14) T F If $f$ and $g$ have the same trace $\{x = 5\}$ then $f_y(5,y) = g_y(5,y)$ for all $y$.

15) T F The surface area of $\vec{r}_1(u,v) = \langle u\cos(v), u\sin(v), u^2\rangle$ and $\vec{r}_2(u,v) = \langle \sqrt{u}\cos(v), \sqrt{u}\sin(v), u\rangle$ defined on $\{0 \leq u, v \leq 1\}$ are the same.

16) T F If $\vec{r}(t)$ is a curve on a graph $z = f(x,y)$ of a function $f(x,y)$, then the velocity vector of $\vec{r}$ is perpendicular to the vector $\langle f_x, f_y, -1\rangle$.

17) T F A continuous function $f(x,y)$ on the closed disc $R = \{x^2 + y^2 \leq 51^2\}$ (of course, $R$ is called **"area** $51\pi$**"**) has a global maximum on $R$.

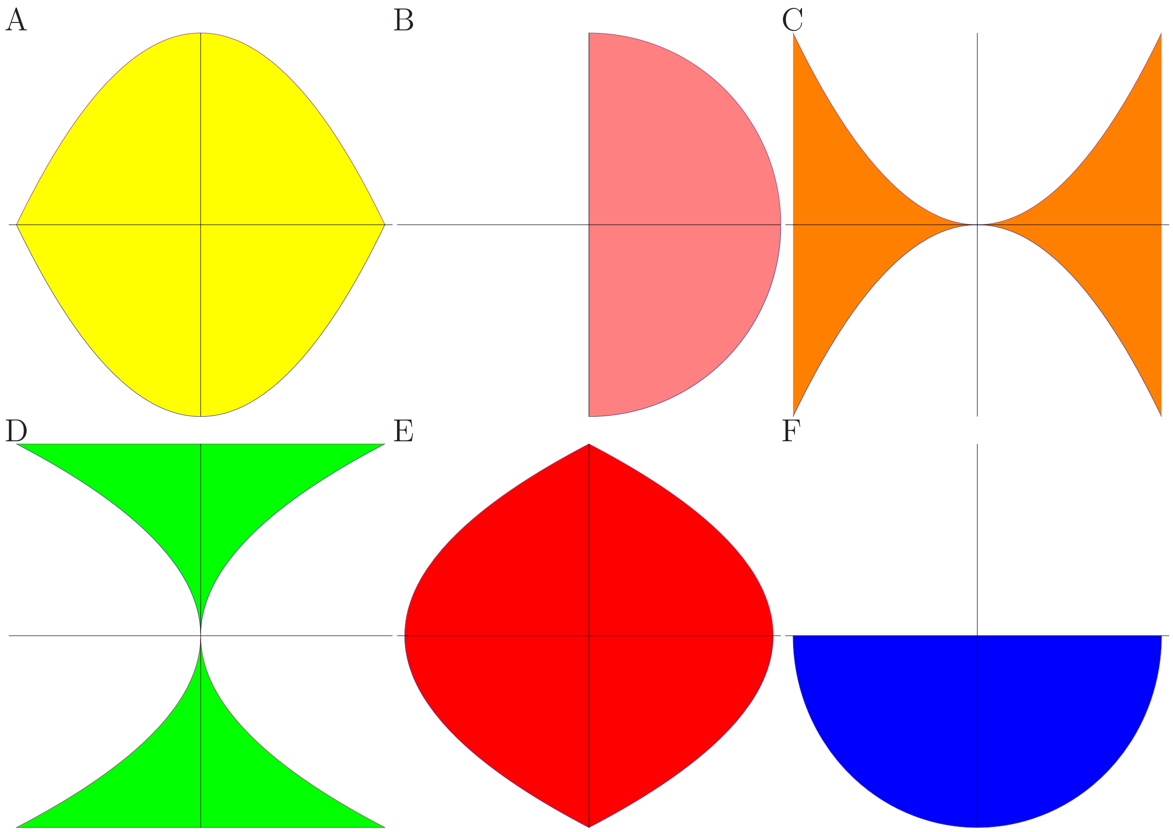
18) T F Any continuous function $f(x,y)$ has a global minimum and maximum on the curve $y = x^2$.

19) T F Fubini's theorem assures that $\int_a^b \int_c^d f(x,y)\, dydx = \int_a^b \int_c^d f(x,y)\, dxdy$.

20) T F $\int\int_R \sin(x+y)\, dxdy = 0$ for $R = \{-\pi \leq x \leq \pi, -\pi \leq y \leq \pi\}$.

Problem 2) (10 points)

a) (6 points) Match the integration regions with the integrals. Each integral matches exactly one region $A - F$.




| Enter A-F | Integral |
|---|---|
| | $\int_{-1}^{1} \int_{-x^2}^{x^2} f(x,y)\,dydx.$ |
| | $\int_{-1}^{1} \int_{-y^2}^{y^2} f(x,y)\,dxdy.$ |
| | $\int_{-1}^{1} \int_{y^2-1}^{1-y^2} f(x,y)\,dxdy.$ |
| | $\int_{-1}^{1} \int_{0}^{\sqrt{1-y^2}} f(x,y)\,dxdy.$ |
| | $\int_{-1}^{1} \int_{x^2-1}^{1-x^2} f(x,y)\,dydx.$ |
| | $\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{0} f(x,y)\,dydx.$ |

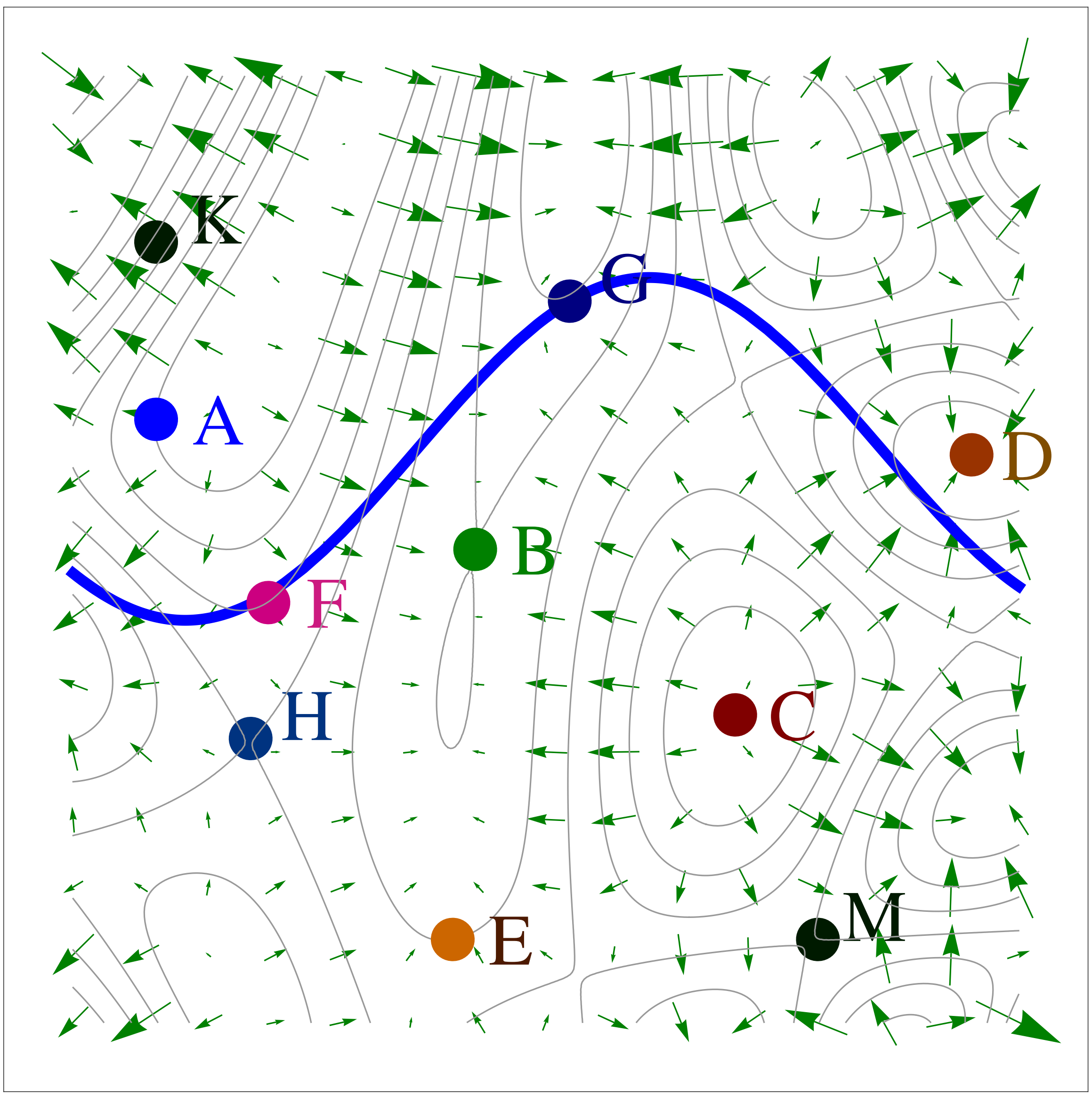
b) (4 points) Fill in one word names (like "Heat","Wave" etc) for the partial differential equations:

| Enter one word | PDE |
|---|---|
| | $g_x = g_y$ |
| | $g_{xx} = g_{yy}$ |
| | $g_{xx} = -g_{yy}$ |
| | $g_x = g_{yy}$ |

Problem 3) (10 points)

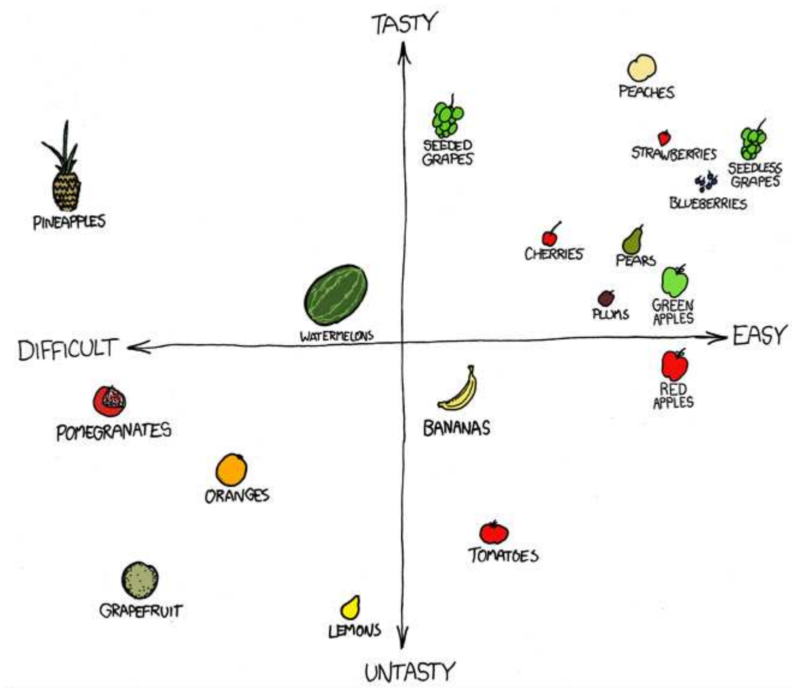
(10 points) A function $f(x, y)$ of two variables has level curves as shown in the picture. We also see a constraint in the form of a curve $g(x, y) = 0$ which has the shape of the graph of the cos function. The arrows show the gradient. In this problem, each of the 10 letters $A, B, C, D, E, F, G, H, K, M$ appears exactly once.

| Enter A-P | Description |
|---|---|
| | a local maximum of $f(x, y)$. |
| | a local minimum of $f(x, y)$. |
| | a saddle point of $f(x, y)$ where $f_{xx} < 0$. |
| | a saddle point of $f(x, y)$ where $f_{xx} > 0$. |
| | a saddle point of $f(x, y)$ where $f_{xx}$ is close to zero |
| | a point, where $f_x = 0$ and $f_y \neq 0$ |
| | a point, where $f_y = 0$ and $f_x \neq 0$ |
| | the point, where $\|\nabla f\|$ is largest |
| | a local maximum of $f(x, y)$ under the constraint $g(x, y) = 0$. |
| | a local minimum of $f(x, y)$ under the constraint $g(x, y) = 0$. |



Problem 4) (10 points)

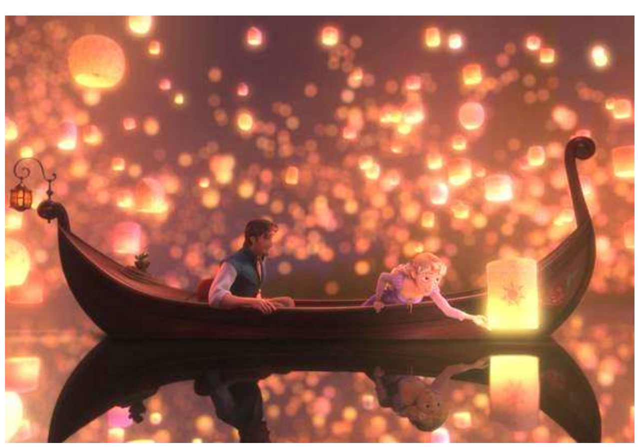
Find and classify all the extrema of the function $f(x, y) = x^5 + y^3 - 5x - 3y$. This function measures "eat temptation" in the x=Easy-y=Tasty plane. Is there a global minimum or global maximum?

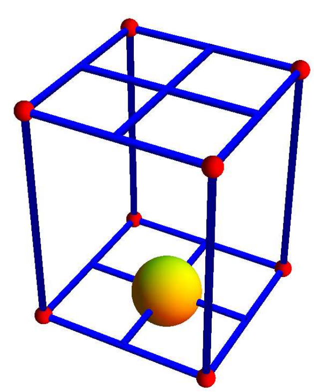



The "Easy-Tasty plane" was introduced in the XKCD cartoon titled "F&#% Grapefruits".

Problem 5) (10 points)

After having watched the latest Disney movie "Tangled", we want to build a hot air balloon with a cuboid mesh of dimension $x, y, z$ which together with the top and bottom fortifications uses wires of total length $g(x, y, z) = 6x + 6y + 4z = 32$. Find the balloon with maximal volume $f(x, y, z) = xyz$.





Problem 6) (10 points)

a) (8 points) Find the tangent plane to the surface $f(x,y,z) = x^2 - y^2 + z = 6$ at the point $(2,1,3)$.

b) (2 points) A curve $\vec{r}(t)$ on that tangent plane of the function $f(x,y,z)$ in a) has constant speed $|\vec{r}\,'| = 1$ and passes through the point $(2,1,3)$ at $t = 0$. What is $\frac{d}{dt}f(\vec{r}(t))$ at $t = 0$?

Problem 7) (10 points)

a) (5 points) Estimate $\sqrt{\sin(0.0004) + 1.001^2}$ using linear approximation.

b) (5 points) We know $f(0,0) = 1$, $D_{\langle \frac{3}{5}, \frac{4}{5} \rangle} f(0,0) = 2$ and $D_{\langle -\frac{4}{5}, \frac{3}{5} \rangle} f(0,0) = -1$. If $L(x,y)$ is the linear approximation to $f(x,y)$ at the point $(0,0)$, find $L(0.06, 0.08)$.

Problem 8) (10 points)

a) (5 points) Find the following double integral

$$\int_0^1 \int_{x^2}^{\sqrt{x}} \frac{\pi \sin(\pi y)}{y^2 - \sqrt{y}} \, dydx \; .$$

b) (5 points) Evaluate the following double integral

$$\int \int_R \frac{\sin(\pi \sqrt{x^2 + y^2})}{\sqrt{x^2 + y^2}} \, dxdy$$

over the region

$$R = \{x^2 + y^2 \leq 1, x > 0 \} \; .$$

Problem 9) (10 points)

a) (8 points) Find the surface area of the surface parametrized as

$$\vec{r}(u,v) = \langle u - v, u + v, (u^2 - v^2)/2 \rangle \; ,$$

where $(u,v)$ is in the unit disc $R = \{u^2 + v^2 \leq 1 \}$.

b) (2 points) Give a nonzero vector $\vec{n}$ normal to the surface at $\vec{r}(4,2) = \langle 2, 6, 6 \rangle$.

Problem 10) (10 points)

a) (6 points) Integrate

$$\int_0^{\pi/2} \int_x^{\pi/2} \frac{\cos(y)}{y} \, dydx$$

b) (4 points) Find the moment of inertia

$$\iint_R (x^2 + y^2) \, dydx \; ,$$

where $R$ is the ring $1 \leq x^2 + y^2 \leq 9$.