Name:

MWF 9 Jameel Al-Aidroos
MWF 9 Dennis Tseng
MWF 10 Yu-Wei Fan
MWF 10 Koji Shimizu
MWF 11 Oliver Knill
MWF 11 Chenglong Yu
MWF 12 Stepan Paul
TTH 10 Matt Demers
TTH 10 Jun-Hou Fung
TTH 10 Peter Smillie
TTH 11:30 Aukosh Jagannath
TTH 11:30 Sebastian Vasey

- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3,8, we need to see **details** of your computation.
- All functions can be differentiated arbitrarily often unless otherwise specified.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

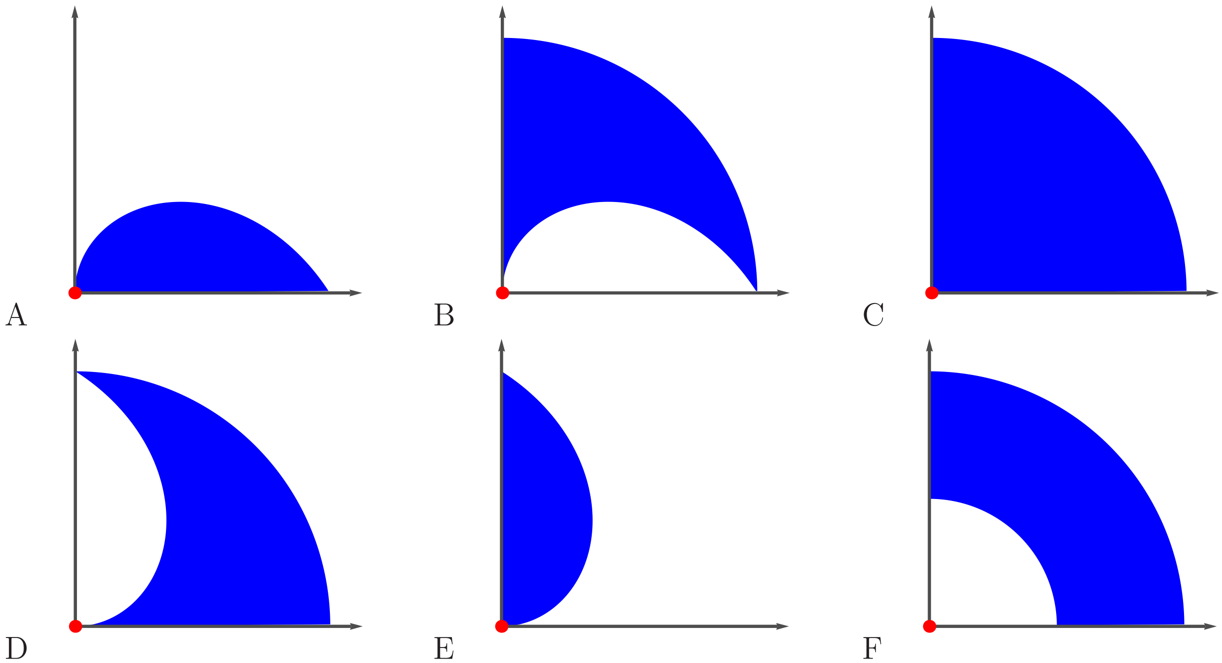
1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) True/False questions (20 points), no justifications needed

1)	T	F	If $f(x,y)=1$ is a curve, and near $(2,3)$ one can write y as a function of x, then $y' = -f_y(2,3)/f_x(2,3)$.
2)	T	F	If $\iint_R f(x,y)\ dA = 0$, then the function $f(x,y)$ is everywhere zero on $R = \{x^2+y^2 \leq 1\}$.
3)	T	F	The directional derivative in the direction of the gradient is $\lvert\nabla f\rvert$.
4)	T	F	The linearization of $f(x,y) = x^3+y^3$ at $(1,1)$ is the quadratic function $L(x,y) = 3x^2+3y^2$.
5)	T	F	The function $f(x,y) = x^2+y^2$ satisfies the partial differential equation $D = f_{xx}f_{yy} - f_{xy}^2 = 1$.
6)	T	F	The function x^2y^2 has no local minimum at $(0,0)$ because the discriminant function D is zero there.
7)	T	F	The double integral $\int_0^{\pi/4}\int_0^2 r^3\ drd\theta$ is the volume of the part of a solid cylinder $x^2+y^2 \leq 4$ which is below the paraboloid $z = x^2+y^2$ and above the xy plane.
8)	T	F	The gradient of $f(x,y,z)$ at (x_0,y_0,z_0) is perpendicular to the level surface of f through (x_0,y_0,z_0).
9)	T	F	If $f(x,y,z) = 3x-4z$, then the minimal possible directional derivative $D_{\vec{u}}f$ at any point in space is -5.
10)	T	F	If (x,y) is not a critical point, then the directional derivative $D_{\vec{v}}f$ can take both positive and negative values for different choices of $\vec{v}$.
11)	T	F	Using linearization of $f(x,y) = x/y$ we can estimate $1.01/1.001 = f(1.01,1.001) \sim 1+0.01-0.001 = 1.009$.
12)	T	F	If $(0,0)$ is a critical point of $f(x,y)$ with nonzero discriminant $D = f_{xx}f_{yy} - f_{xy}^2$, we know that it is either a saddle, a global maximum or a global minimum.
13)	T	F	For a rectangular region R, Fubini tells that $\int_0^2\int_0^3 f(x,y)\ dxdy = \int_0^2\int_0^3 f(x,y)\ dydx$ for any continuous function $f(x,y)$.
14)	T	F	If a function $f(x,y)$ has only one critical point $(0,0)$ in $G = \{x^2+y^2 \leq 1\}$ which is a local maximum and $f(0,0)=1$, then $\iint_G f(x,y)\ dxdy > 0$.
15)	T	F	If $\vec{r}(t)$ is a curve in space for which the speed is 1 at all times and $f(x,y,z)$ is a function of three variables, then $d/dt f(\vec{r}(t)) = D_{\vec{r}\,'(t)}(f)$.
16)	T	F	$\int_0^1\int_0^1 f_{xy}(x,y)\ dydx = f(1,1) - f(1,0) - f(0,1) + f(0,0)$.
17)	T	F	If $f_{yy}(x,y) > 0$ everywhere, then f can not have any local maximum.
18)	T	F	The double integral $\int_0^1\int_0^1 x^2 - y^2\ dxdy$ is the volume of the solid below the graph of $f(x,y) = x^2-y^2$ and above the square $0 \leq x \leq 1, 0 \leq y \leq 1$ in the xy-plane.
19)	T	F	For any unit vector $\vec{v}$ and any differentiable function f, one has $D_{\vec{v}}(f) + D_{-\vec{v}}(f) = 0$.
20)	T	F	The surfaces $x+y+z=0$ and $x^2+y^2+z^2+x+y+z=0$ have the same tangent plane at $(0,0,0)$.

Problem 2) (10 points)

a) (6 points) Match the regions with the corresponding polar double integrals



Enter A-F	Integral of $f(r,\theta)$
	$\int_0^{\pi/2}\int_0^{\pi/2} f(r,\theta)r\ drd\theta$
	$\int_0^{\pi/2}\int_0^{\theta} f(r,\theta)r\ drd\theta$
	$\int_0^{\pi/2}\int_0^{\pi/2-\theta} f(r,\theta)r\ drd\theta$

Enter A-F	Integral of $f(r,\theta)$
	$\int_0^{\pi/2}\int_{\theta}^{\pi/2} f(r,\theta)r\ drd\theta$
	$\int_0^{\pi/2}\int_{\pi/2-\theta}^{\pi/2} f(r,\theta)r\ drd\theta$
	$\int_0^{\pi/2}\int_{\pi/4}^{\pi/2} f(r,\theta)r\ drd\theta$

b) (4 points) Match the partial differential equations (PDE's) for the functions $u(t,s)$ with their names. No justifications are needed.

Enter A,B,C,D here	PDE
	$u_t + uu_s - u_{ss} = 0$
	$u_{tt} + u_{ss} = 0$

Enter A,B,C,D here	PDE
	$u_{tt} - u_{ss} = 0$
	$u_t - u_{ss} = 0$

A) Wave equation	B) Heat equation	C) Burgers equation	D) Laplace equation

Problem 3) (10 points)

a) (7 points) Find and classify all the critical points of the function

$$f(x,y) = 5 + 3x^2 + 3y^2 + y^3 + x^3 \ .$$

b) (3 points) Is there a global maximum or a global minimum for $f(x,y)$?

Problem 4) (10 points)

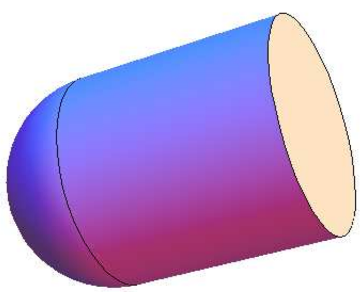
A **solid bullet** made of a half sphere and a cylinder has the volume $V = 2\pi r^3/3 + \pi r^2 h$ and surface area $A = 2\pi r^2 + 2\pi r h + \pi r^2$. Doctor Manhatten designs a bullet with fixed volume and minimal area. With $g = 3V/\pi = 1$ and $f = A/\pi$ he therefore minimizes

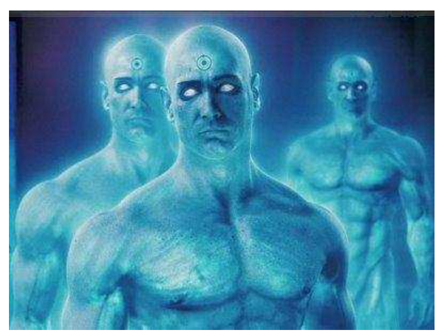
$$f(h,r) = 3r^2 + 2rh$$

under the constraint

$$g(h,r) = 2r^3 + 3r^2 h = 1 \ .$$

Use the Lagrange method to find a local minimum of f under the constraint $g = 1$.



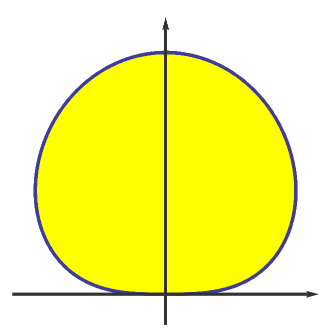


Problem 5) (10 points)

A region R in the plane shown to the right is called the **"blob of nothingness"**. It does not have any purpose nor meaning. It just sits there. The region is given in polar coordinates as $0 \leq r \leq \theta(\pi - \theta)$ for $0 \leq \theta \leq \pi$. Find the area

$$\int \int_R 1 \ dxdy$$

of this nihilistic object.



Problem 6) (10 points)

a) (4 points) If

$$f(x, y) = y \cos(x - y) ,$$

find equation of plane tangent to $z = f(x, y)$ at the point $(2, 2, 2)$.

b) (3 points) Find the equation of the tangent line to $f(x, y) = 2$ at $(2, 2)$.

c) (3 points) Estimate $f(2.1, 1.9)$ using linear approximation.

Problem 7) (10 points)

A **Harvard robot bee** flies along the curve

$$\vec{r}(t) = \langle t - t^3, 3t^2 - 3t \rangle$$

and measures the temperature $f(x, y)$. It flies over the target point $(0, 0)$ at time $t = 0$ and time $t = 1$. At each time, its sensor measures the temperature change $g'(t)$ where $g(t) = f(\vec{r}(t))$.

Image source: Harvard Press release on robobees.seas.harvard.edu

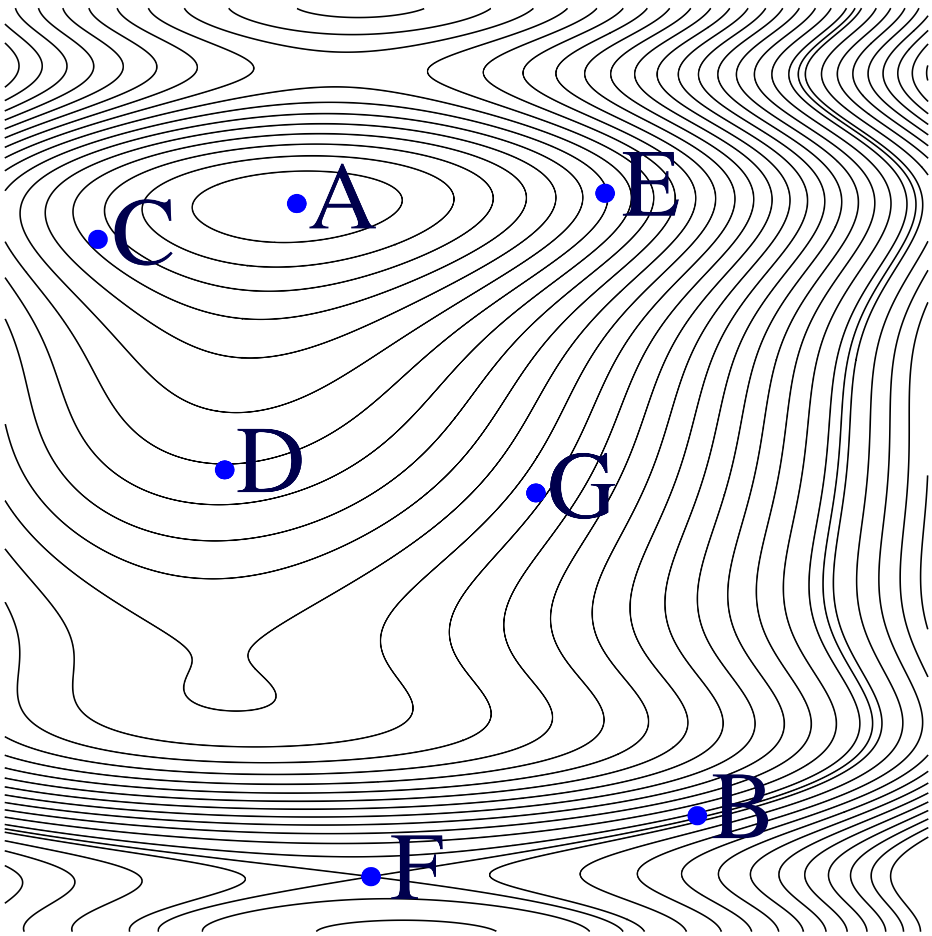
a) (5 points) Assume you knew that the gradient of f at $(0, 0)$ is $\langle a, b \rangle$. What are the values of $g'(t) = d/dt f(\vec{r}(t))$ at $t = 0$ and $t = 1$ in terms of a and b?

b) (5 points) The bee measures $g'(0) = 3$ and $g'(1) = 3$. What is the gradient $\nabla f(0, 0) = \langle a, b \rangle$ of f at $(0, 0)$?

Problem 8) (10 points)

A function $f(x, y)$ of two variables has level curves as shown in the picture. The function values at neighboring level curves differ by 1. [No justifications are needed in this problem. Naturally, since there are less points then boxes, some of the points A-G will appear more than once, but each box will only be filled with one letter.]

Enter A-G	is a point, where ...
	$f_x(x,y) = 0$ and $f_y(x,y) \neq 0$.
	$f_y(x,y) = 0$ and $f_x(x,y) \neq 0$.
	$f(x,y)$ has either a max or a min.
	$f(x,y)$ has a saddle point.
	$f(x,y)$ has no max nor min but is extremal under a constraint $y = c$ for some c.
	$f(x,y)$ has no max nor min but is extremal under a constraint $x = c$ for some c.
	the length of the gradient vector of f is largest among all points A-G.
	$D_{\langle 1/\sqrt{2}, 1/\sqrt{2}\rangle} f(x,y) = 0$ and $D_{\langle 1/\sqrt{2}, -1/\sqrt{2}\rangle} f(x,y) \neq 0$.
	$D_{\langle 1/\sqrt{2}, -1/\sqrt{2}\rangle} f(x,y) = 0$ and $D_{\langle 1/\sqrt{2}, 1/\sqrt{2}\rangle} f(x,y) \neq 0$.
	the tangent line to the curve is $x + y = d$ for some constant d.



Problem 9) (10 points)

Evaluate the following double integral

$$\int_0^1 \int_0^{(1-x)^2} \frac{x^3}{(1-\sqrt{y})^4} \, dydx \, .$$

Problem 10) (10 points)

A mass point with position (x, y) is attached by springs to the points $A_1 = (0, 0), A_2 = (2, 0), A_3 = (0, 2), A_4 = (2, 3), A_5 = (3, 1)$. It has the potential energy

$$f(x, y) = 31 - 14x + 5x^2 - 12y + 5y^2$$

which is the sum of the squares of the distances from (x, y) to the 5 points. Find all extrema of f using the second derivative test. The minimum of f is the position, where the mass point has the lowest energy.