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- Start by printing your name in the above box and **check your section** in the box to the left.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- **Show your work**. Except for problems 1-3, we need to see details of your computation.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 180 minutes time to complete your work.

1		20
2		10
3		10
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11		10
12		10
13		10
14		10
Total:		150

Problem 1) True/False questions (20 points)

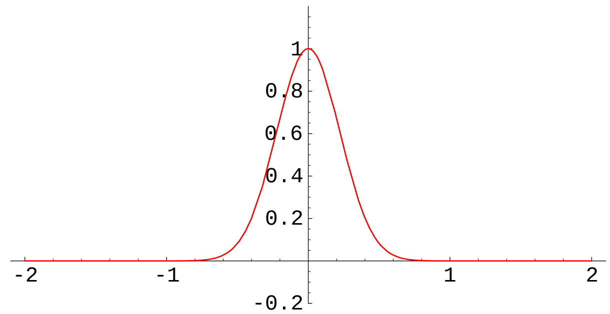
1)	T	F	The distance from $(1,2,-1)$ to $(3,-2,1)$ is $(-2,4,-2)$.
2)	T	F	The plane $y=3$ is perpendicular to the xz plane.
3)	T	F	All functions $u(x,y)$ that obey $u_x = u$ at all points obey $u_y = 0$ at all points.
4)	T	F	The best linear approximation at $(1,1,1)$ to the function $f(x,y,z) = x^3 + y^3 + z^3$ is the function $L(x,y,z) = 3x^2 + 3y^2 + 3z^2$
5)	T	F	If $f(x,y)$ is any function of two variables, then $\int_0^1 \left(\int_x^1 f(x,y)\, dy\right) dx = \int_0^1 \left(\int_y^1 f(x,y)\, dx\right) dy$.
6)	T	F	Let $C = \{(x,y) \mid x^2+y^2 = 1\}$ be the unit circle in the plane and $\vec{F}(x,y)$ a vector field satisfying $\lvert\vec{F}\rvert \leq 1$. Then $-2\pi \leq \int_C \vec{F} \cdot dr \leq 2\pi$.
7)	T	F	Let $\vec{a}$ and $\vec{b}$ be two nonzero vectors. Then the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ always point in different directions.
8)	T	F	If all the second-order partial derivatives of $f(x,y)$ vanish at (x_0,y_0) then (x_0,y_0) is a critical point of f.
9)	T	F	If $\vec{a},\vec{b}$ are vectors, then $\lvert\vec{a}\times\vec{b}\rvert$ is the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$.
10)	T	F	The distance between two points A,B in space is the length of the curve $\vec{r}(t) = A + t(B-A),\ t \in [0,1]$.
11)	T	F	The function $f(x,y) = xy$ has no critical point.
12)	T	F	The length of a curve does not depend on the chosen parameterization.
13)	T	F	There exists a non-zero function $f(x,y,z)$ and non-zero vector field $\vec{F}(x,y,z)$ so that $\vec{F} = \text{grad}(f)$ and $f = \text{div}\vec{F}$.
14)	T	F	For any numbers a,b satisfying $\lvert a\rvert \neq \lvert b\rvert$, the vector $\langle a-b, a+b\rangle$ is perpendicular to $\langle a+b, b-a\rangle$.
15)	T	F	The line integral of $\vec{F}(x,y) = \langle -y, x\rangle$ along the counterclockwise oriented boundary of a region R is twice the area of R.
16)	T	F	There is no surface for which both the parabola and the hyperbola appear as traces.
17)	T	F	If $(u,v) \mapsto \vec{r}(u,v)$ is a parameterization for a surface, then $\vec{r}_u(u,v) + \vec{r}_v(u,v)$ is a vector which lies in the tangent plane to the surface.
18)	T	F	When using spherical coordinates in a triple integral, one needs to include the volume element $dV = \rho^2 \cos(\phi)\, d\rho d\phi d\theta$.
19)	T	F	A surface in space for which all normal vectors are parallel to each other must be part of a plane.
20)	T	F	A vector field $\vec{F} = \langle P(x,y), Q(x,y)\rangle$ is conservative in the plane if and only if $P_y(x,y) = Q_x(x,y)$ for all points (x,y).

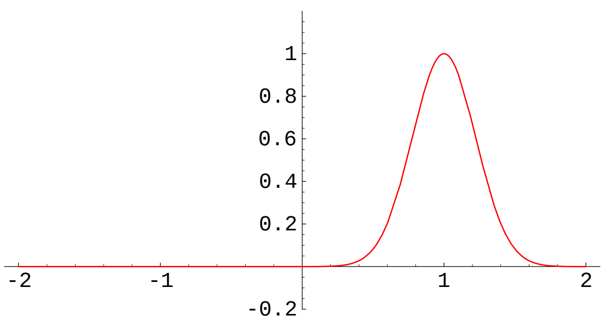
Problem 2) (10 points)

2 a) (5 points) Fill in names of the mathematicians: Green, Stokes, Gauss, Fubini, Clairot. If there is no name associated to the theorem, write the name of the theorem.

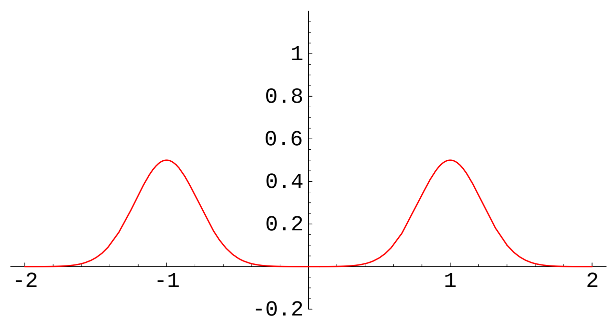
Formula	Name of the theorem
$\int_C \vec{F} \cdot d\vec{r} = \int \int_S \text{curl}(\vec{F}) \cdot dS$	
$f_{xy}(x,y) = f_{yx}(x,y)$	
$\int_C \vec{F} \cdot dr = \int \int_R \text{curl}(\vec{F})\ dxdy$	
$\int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t))\ dt = f(\vec{r}(b)) - f(\vec{r}(a))$	
$\int \int_S F \cdot dS = \int \int \int_E \text{div}(F)\ dV$	
$\int_a^b \int_c^d f(x,y)\ dxdy = \int_c^d \int_a^b f(x,y)\ dydx$	

2 b) (5 points) We have a function $u(t,x)$ which is a solution to a partial differential equation. In all cases, we have $u(0,x) = e^{-x^2}$. The picture to the right shows this function $u(0,x)$. Which partial differential equation is involved, when you see the function $u(1,x)$ as a graph?

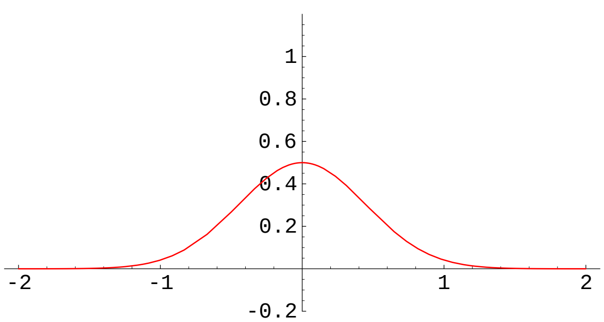




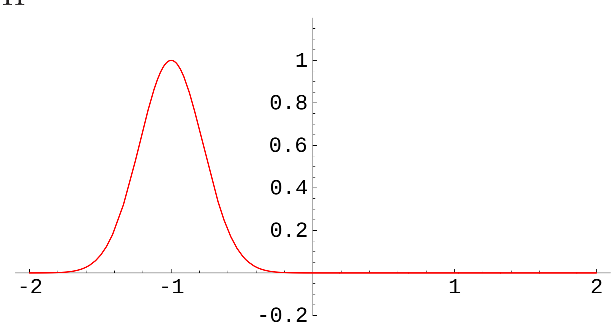
I



II



III



IV

Enter I,II,III,IV here	Equation
	$u_t(x,t) = u_x(x,t)$
	$u_t(x,t) = u_{xx}(x,t)$
	$u_{tt}(x,t) = u_{xx}(x,t)$
	$u_t(x,t) = -u_x(x,t)$

Problem 3) (10 points)

a) Find an equation for the plane Σ passing through the points $P = (1, 0, 1)$, $Q = (2, 1, 3)$ and $R = (0, 1, 5)$.

b) Find the distance from the origin $O = (0, 0, 0)$ to Σ.

c) Find the distance from the point P to the line through Q, R.

d) Find the volume of the parallelepiped with vertices O, P, Q, R.

Problem 4) (10 points)

The equation $f(x, y, z) = e^{xyz} + z = 1 + e$ implicitly defines z as a function $z = g(x, y)$ of x and y.

a) Find formulas (in terms of x,y and z) for $g_x(x, y)$ and $g_y(x, y)$.

b) Estimate $g(1.01, 0.99)$ using linear approximation.

Problem 5) (10 points)

Find the surface area of the surface S parametrized by $\vec{r}(u, v) = \langle u, v, 2 + \frac{u^2}{2} + \frac{v^2}{2} \rangle$ for (u, v) in the disc $D = \{u^2 + v^2 \leq 1\}$.

Problem 6) (10 points)

Find the local and global extrema of the function $f(x, y)$ which is the curl of $\vec{F}(x, y) = \langle -y^4/12 + y^3/6 - y, x^4/12 - x^3/6 \rangle$ on the disc $\{x^2 + y^2 \leq 4\}$.

a) Classify every critical point inside the disc $x^2 + y^2 < 4$.

b) Find the extrema on the boundary $\{x^2 + y^2 = 4\}$ using the method of Lagrange multipliers.

c) Determine the global maxima and minima on all of D.

Problem 7) (10 points)

a) Given two nonzero vectors $\vec{u} = \langle a, b, c \rangle$ and $\vec{v} = \langle d, e, f \rangle$ in space, write down a formula for the cosine of the angle between them. Find a nonzero vector $\vec{v}$ that is perpendicular to $\vec{u} = \langle 3, 2, 1 \rangle$. Describe geometrically the set of all $\vec{v}$, including zero, that are perpendicular to this vector $\vec{u}$.

b) Consider a function f of three variables. Explain with a picture and a sentence what it means geometrically that $\nabla f(P)$ is perpendicular to the level set of f through P.

c) Assume the gradient of f at P is nonzero. Write a few sentences that would convince a skeptic that $\nabla f(P)$ is perpendicular to the level set of f at the point P.

d) Assume the level set of f is the graph of a function $g(x, y)$. Explain the relation between the gradient of g and the gradient of f. Especially, how do you relate the orthogonality of ∇f to the level set of f with the orthogonality of ∇g to the level set of g?

Problem 8) (10 points)

Let R be the region inside the circle $x^2 + y^2 = 4$ and above the line $y = \sqrt{3}$. Evaluate

$$\int\int_R \frac{y}{x^2 + y^2} \, dA \, .$$

Problem 9) (10 points)

A region W in $\mathbf{R}^3$ is given by the relations

$$x^2 + y^2 \leq z^2 \leq 3(x^2 + y^2)$$
$$1 \leq x^2 + y^2 + z^2 \leq 4$$
$$x \geq 0$$

1. Sketch the region W.
2. Find the volume of the region W.

Problem 10) (10 points)

Consider the vector field

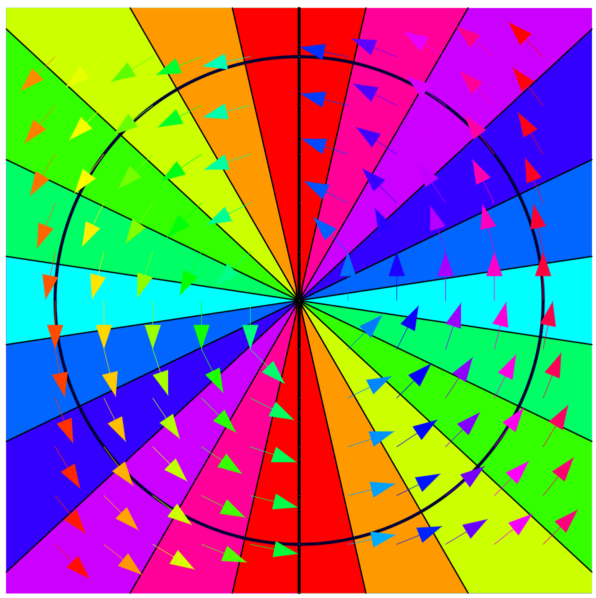
$$\vec{F}(x,y) = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle$$

defined everywhere in the plane $\mathbf{R}^2$ except at the origin.

a) Let C be any closed curve which bounds a region D. Assume that $(0,0)$ is not contained in D and does not lie on C. Explain why

$$\int_C \vec{F} \cdot d\vec{r} = 0 \ .$$

b) Let C be the unit circle oriented counterclockwise. What is $\int_C \vec{F} \cdot d\vec{r}$? Explain why your answer shows that there is no function f for which $\vec{F}(x,y) = \nabla f(x,y)$ everywhere.



Problem 11) (10 points)

First use rectangular, then cylindrical and finally spherical coordinates to integrate the function $f(x,y,z) = xyz$ over the solid in space described by the inequalities $0 \leq z \leq \sqrt{(1-x^2-y^2)}, x^2+y^2 \leq 1, x-y \geq 0, y \geq 0$.

Problem 12) (10 points)

Let $\vec{F}(x,y)$ be a vector field in the plane given by the formula

$$\vec{F}(x,y) = \langle x^2 - 2xye^{-x^2} + 2y, e^{-x^2} + \frac{1}{\sqrt{y^4+1}} \rangle .$$

If C is the path which goes from from $(-1,0)$ to $(1,0)$ along the semi circle $x^2 + y^2 = 1$, $y \geq 0$, evaluate $\int_C \vec{F} \cdot d\vec{r}$.

Problem 13) (10 points)

In appropriate units, the charge density $\sigma(x,y,z)$ in a region in space is given by $\sigma = \nabla \cdot \vec{E} = \text{div}(\vec{E})$, where $\vec{E}$ is the electric field. Consider the cube of side lengths 1 given by $0 \leq x, y, z \leq 1$. What is the total charge in this cube if

$$\vec{E} = \langle x(1-x)\log(1+xyz), y(1-y)\tan(x^3+y^3+z^3), z(1-z)e^{\sqrt{x+y}} \rangle .$$

(The total charge is the integral of the charge density over the cube.)

Problem 14) (10 points)

a) By calculating the integral $\int\int_S \vec{F} \cdot \vec{dS}$ directly, find the flux of the vector field $\vec{F}(x,y,z) = \langle 0, 0, x+z \rangle$ through the sphere $x^2 + y^2 + z^2 = 9$, where the sphere is oriented with the normal pointing outward.

b) Find the flux of the vector field $\vec{F}(x,y,z) = \langle 0, 0, x+z \rangle$ through the sphere $x^2+y^2+z^2 = 9$ using the divergence theorem.

c) Explain in words without invoking any integral theorem, why the flux integral of the vector field $\vec{F}(x,y,z) = \langle 0, 0, x+z \rangle$ through any sphere with positive radius centered at $(0,0,0)$ is positive. A one or two sentence explanation is sufficient, but it should be formulated so that it makes sense to somebody who does not know calculus.