Homework 10: Partial derivatives

This homework is due Monday, 10/2 rsp Tuesday 10/3.

1. If \( f(x, y) = \sqrt{4 - x^2 + 4y^2} \), find \( f_x(1, 2) \) and \( f_y(1, 2) \) and interpret these numbers as slopes. Illustrate this with either hand-drawn sketches of traces (functions of one variable) or using computer plots.

2. Find the partial derivatives \( f_x(x, y) \), \( f_y(x, y) \) of the function \( f(x, y) = 5xy \) at the point \( (1, 2) \).

3. Find the first partial derivatives \( f_x(x, y) \), \( f_y(x, y) \) of the function

\[
 f(x, y) = \int_y^x \sin(t^2) \, dt .
\]

4. Verify that for any constant \( k \) and \( \alpha \), the function

\[
 u(x, t) = e^{-\alpha^2k^2t} \sin(kx)
\]

is a solution of the heat equation \( u_t(x, t) = \alpha^2 u_{xx}(x, t) \).

5. Verify that the function

\[
 u(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}
\]

is a solution of the three dimensional Laplace equation \( u_{xx} + u_{yy} + u_{zz} = 0 \).
Main definitions

If \( f(x, y) \) is a function of two variables, then \( \frac{\partial}{\partial x} f(x, y) \) is defined as the derivative of the function \( g(x) = f(x, y) \), where \( y \) is considered a constant. It is called **partial derivative** of \( f \) with respect to \( x \). The partial derivative with respect to \( y \) is defined similarly. We also write \( f_x(x, y) = \frac{\partial}{\partial x} f(x, y) \) and \( f_{yx} = \frac{\partial}{\partial x} \frac{\partial}{\partial y} f \).

**Clairaut’s theorem** If \( f_{xy} \) and \( f_{yx} \) are both continuous, then \( f_{xy} = f_{yx} \).

We will in the next lecture look more at partial differential equations: an equation for an unknown function \( f(x, y) \) which involves partial derivatives with respect to at least two different variables is called a **partial differential equation**. If only the derivative with respect to one variable appears, it is called an **ordinary differential equation**. Both ordinary and partial differential equations are of great importance in other sciences. Here are examples we are going to look in the next class:

1. The **wave equation** \( f_{tt}(t, x) = f_{xx}(t, x) \) governs the motion of light or sound.
2. The **heat equation** \( f_t(t, x) = f_{xx}(t, x) \) describes diffusion of heat or spread of an epidemic.
3. The **Laplace equation** \( f_{xx} + f_{yy} = 0 \) determines the shape of a membrane.
4. The **advection equation** \( f_t = f_x \) is used to model transport in a wire.
5. The **eiconal equation** \( f_x^2 + f_y^2 = 1 \) is used to see the evolution of wave fronts in optics.
6. The **Burgers equation** \( f_t + f f_x = f_{xx} \) describes waves at the beach which break.