Homework 6: Matrix Algebra

This homework is due on Friday, February 9, respectively on Tuesday February 13, 2018.

1 For each pair of matrices $A$ and $B$, compute both $AB$ and $BA$
   a) $A = \begin{bmatrix} 2 & 3 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 2 \\ -1 & 3 \end{bmatrix}$.
   b) $A = \begin{bmatrix} 2 & 3 & 4 \\ 2 & -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 2 \\ 2 & 3 \\ 6 & 11 \end{bmatrix}$.

Solution:
   a) $AB = \begin{bmatrix} 1 & 13 \\ 0 & 16 \end{bmatrix}$, $BA = \begin{bmatrix} 8 & 14 \\ 4 & 9 \end{bmatrix}$.

   b) $AB = \begin{bmatrix} 38 & 57 \\ 6 & 1 \end{bmatrix}$.
      
      $BA = \begin{bmatrix} 12 & 10 & 16 \\ 10 & 3 & 8 \\ 34 & 7 & 24 \end{bmatrix}$.

2 a) Find a $2 \times 2$ matrix $A$ with no 0 or 2 entries such that $A^2 = 0$.
   b) Can you find a $3 \times 3$ matrix $A$ with entries $-2, 1$ such that $A^2 = 0$? (You can search computer assisted).
   c) Can you find a 4 times 4 matrix with entries $-1, 3$ such that $A^2 = 0$?

(* See what this does and modify *)

Do[A=Table[RandomChoice[{-1,1}],{4},{4}];
   If[Max[A.A]==0,Print[A],{10000}]]
Solution:

a) \( A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \) works (note that this is because the rows are orthogonal to both columns).  

b) An example is 
\[
\begin{bmatrix}
1 & 1 & 1 \\
-2 & -2 & -2 \\
1 & 1 & 1
\end{bmatrix}
\]

c) An example is 
\[
\begin{bmatrix}
-1 & -1 & -1 & -1 \\
3 & 3 & 3 & -1 \\
-1 & 3 & 3 & -1 \\
3 & 3 & -1 & 3
\end{bmatrix}
\]

3  a) Find the inverse of the matrix \( A \) made from the first 4 rows of Pascal’s triangle. \( A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} \).

b) The following 0–1 matrix \( B \) has the property that the inverse is again an integer matrix: 
\( B = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \). Find the inverse.
Solution:

a) \( A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \).

b) \( B^{-1} = \begin{bmatrix} -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & -1 & 0 \end{bmatrix} \).

4 a) Assume \( A^8 = A \cdot A \cdot A \cdot A \cdot A \cdot A \cdot A \cdot A \) is the identity matrix. Can you find a simple formula in terms of \( A \) which gives \( A^{-1} \)?
b) Find a transformation in the plane with \( A^4 \) not being the identity such that \( A^8 = 1 \). What is \( A^{-1} \)?

Solution:

a) It is \( A^7 \), because \( A(A^7) = (A^7)A = 1 \).
b) Rotation by \( 2\pi/8 \). We can write down the matrix explicitly: \( A = \begin{bmatrix} \cos(2\pi/8) & -\sin(2\pi/8) \\ \sin(2\pi/8) & \cos(2\pi/8) \end{bmatrix} \). Because of how rotations work, \( A^6 \) is rotation by \( 12\pi/7 \). This means that \( A^{-1} = \begin{bmatrix} \cos(12\pi/8) & -\sin(12\pi/8) \\ \sin(12\pi/8) & \cos(12\pi/8) \end{bmatrix} = \begin{bmatrix} \cos(2\pi/8) & \sin(2\pi/8) \\ -\sin(2\pi/8) & \cos(2\pi/8) \end{bmatrix} \).

5 a) Assume \( A \) is small enough so that \( B = 1 + A + A^2 + A^3 + .... \) converges. Verify that \( B \) is the inverse of \( 1 - A \). (Leontief).
b) Use Mathematica to plot \( A^{-1} \) for the 100 x 100 matrices defined by \( A_{nm} = n^2 + 1.1m \).
c) Use Mathematica to plot \( A^{-1} \) for the 100 x 100 matrix \( A_{n,m} = \)
\(gcd(n, m)\), the greatest common divisor. (If you can explain this pattern, this might be a research paper. It seems unexplored).

**Solution:**

a) By the distributive property, we have that 
\[(1 - A)B = 1B - AB = B - AB = B - A(1 + A + A^2 + \ldots) = B - (A + A^2 + A^3 + \ldots) = B - (B - 1) = 1.\]

b) Use the code below. We see strange stripes.

c) We see rays.

\[
\text{MatrixPlot}\left[\text{Table}[n^2+1.1 \ m,\{n,300\},\{m,300\}]\right];
\]

**Matrix Algebra**

Matrices can be added, multiplied with a scalar. One can also form the product of two matrices \(A \cdot B\) as well as the inverse matrix \(A^{-1}\) if the matrix is invertible. These operations constitute the **matrix algebra**. It behaves like the algebra of real numbers but the multiplication is no more commutative in general. Besides the matrix 0 where all entries are zero there are other matrices which are not invertible. We write 1 for the identity matrix which has 1 in the diagonal and 0 everywhere else. Now \(A1 = A\).

\[
A=\begin{\{\{5,2\},\{3,4\}\}\};
\text{Inverse}[A]+\text{MatrixPower}[A,7]+\text{IdentityMatrix}[2]
\]