Assignment 1 Solutions

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Problem 1.

1. Suppose $e$ is a right identity, i.e. $ge = g$ ($\forall g$). So we have, for any $g_1, g_2 \in G$,
   
   $g_1 g_2 = (g_1 e) g_2 = g_1 (e g_2)$

   Left-multiply by $g_1^{-1}$ to get $g_2 = e g_2$ ($\forall g_2 \in G$).

   Now suppose that some $g \in G$ has a right inverse $g_R^{-1}$ and a left inverse $g_L^{-1}$ that might be different. We have $e = g^{-1} L g$, so
   
   $g_R^{-1} e g_R^{-1} = (g_L^{-1} g) g_R^{-1} = g_L^{-1} (g g_R^{-1}) = g_L^{-1} e = g_L^{-1}$

2. Suppose there are two elements, $e$ and $e'$ which both satisfy $eg = ge = g$ and $e'h = he' = h$ for all $g, h \in G$. Letting $g = e'$ and $h = e$ yields $ee' = e'$ and $ee' = e$, so $e = e'$. Suppose $x, y$ are both inverses of $g$. Then $gx = e = gy$, which implies $x = y$ by left cancellation.

Problem 2. Let $D_a$ and $D_b$ be two finite-dimensional irreducible representations of $G$. Choose a basis $|a, j\rangle$ for the vector space of $D_a$, where $j = 1, \ldots, n_a$ and similarly choose $|b, \ell\rangle$, a basis for the vector space of $D_b$. Define

   $A_{ab}^{j\ell} \equiv \int_G dg D_a(g^{-1})|a, j\rangle \langle b, \ell| D_b(g)$

   Then we have

   $D_a(g_1) A_{ab}^{j\ell} = \int_G dg D_a((gg_1^{-1})^{-1})|a, j\rangle \langle b, \ell| D_b(g) = \int_G dg' D_a(g'^{-1})|a, j\rangle \langle b, \ell| D_b(g'g_1) = A_{ab}^{j\ell} D_b(g_1)$

   By Schur’s Lemma, $\exists$ constants $\lambda_{j\ell}^a$ such that $A_{ab}^{j\ell} = \delta^{ab} \lambda_{j\ell}^a I$. It follows that $Tr_{j\ell}^{ab} = \delta^{ab} \lambda_{j\ell}^a Tr I = \delta^{ab} \lambda_{j\ell}^a n_a$. Also, by cyclicity of the trace,

   $Tr_{j\ell}^{ab} = \delta^{ab} \int_G dg \langle a, \ell| D_a(g) D_a(g^{-1})|a, j\rangle = \delta^{ab} \delta_{j\ell} \int_G dg = \delta^{ab} \delta_{j\ell} \text{vol}(G)$
We have assumed \( \text{vol}(G) = 1 \), therefore \( \lambda_{ij}^a = \delta_{ij}/n_a \) and we have

\[
\int_G dg D_a(g^{-1})(a,j)/\ell(b,\ell|D_b(g) = \frac{\delta_{ab}\delta_{ij}}{n_a} I
\]

Expressed in terms of matrix elements, this can be written as

\[
\int_G dg n_a[D_a(g^{-1})_{kj}[D_b(g)]_{\ell m} = \delta_{ab}\delta_{ij}\delta_{km}
\]

Problem 3.

1. For any \( k \in \mathbb{R} \), let \( D_k(x) = e^{ikx} \), interpreted as a \( 1 \times 1 \) matrix acting on \( \mathbb{C} \). This is clearly a unitary representation for all \( k \in \mathbb{R} \), and any representation on a 1-dimensional vector space is automatically irreducible.

2. Now let \( D_k(x) = e^{kx} \), again interpreted as a \( 1 \times 1 \) matrix acting on \( \mathbb{C} \). A similarity transformation does not change the representation, since all \( 1 \times 1 \) matrices commute. Suppose for some \( k \), \( D_k(x) \) is unitary; that means \( (e^{kx})^{-1} = \overline{e^{kx}} = e^{-kx} \), which is true for all \( x \) only if \( k = -k \), i.e. \( k = 0 \). We conclude that these \( D_k \)'s, when nontrivial, cannot be equivalent to a unitary representation.

3. It is discussed in Georgi, p. 21 that the characters \( \chi_a(g) \) of the independent irreducible representations of a group form a complete orthonormal basis for the space of class functions.\(^1\) Indeed, if \( F \) is a class function, then Eq. (1.85) in Georgi\(^2\) can be interpreted as saying there are constants \( \gamma^a \) such that

\[ F(g) = \sum_a \gamma^a \chi_a(g) \]

The characters of the irreducible unitary reps of the group \( \mathbb{R} \) are \( \chi_a(x) = e^{iax} \), so replacing the sum above by an integral, the analogous statement is that for each \( a \), there is a number \( \gamma(a) \) such that

\[ F(x) = \int_{-\infty}^{\infty} \gamma(a)e^{iax} \, da \]

But the existence of \( \gamma(a) \) is guaranteed precisely by the Fourier inversion theorem. (There is some subtlety here because, in general to apply this, the class function \( F \) must be in the space of square-integrable functions \( L^2(G) \). If \( G \) is finite, then any function is in \( L^2(G) \), but otherwise it is an additional assumption that we have to make.)

4. For finite groups we have the orthogonality relation

\[ \sum_a \chi_a(g_\alpha)^*\chi_a(g_\beta) = \frac{N}{k_\alpha} \delta_{\alpha\beta} \]

where \( \alpha, \beta \) index the distinct conjugacy classes in \( G \), and \( k_\alpha \) = number of elts in the \( \alpha \)'th conjugacy class. For the group \( G = \mathbb{R} \), every conjugacy class has one element so \( \alpha, \beta \) are just

\(^1\)a class function is a function on the group that is constant on conjugacy classes.

\(^2\)In Georgi’s notation, for a finite group \( \gamma^a = \frac{1}{n_a} \sum_j \gamma_j^a \).
real numbers, and the irreducible characters are of the form $\chi_a(x) = e^{i\alpha x}$. We replace $(1/N) \sum_a$ with $\int_G dg$ where $dg$ is assumed to be an invariant measure, and the orthogonality relation becomes

$$\int da e^{-i\alpha x} e^{iay} = \int da e^{i\alpha(y-x)} = 2\pi \delta(y-x)$$