Problem 1 (Electric dipole transitions)

The electric dipole operator is given by $\vec{d}_e = e \vec{r}$. Hence for an electric field in the $\hat{z}$ direction, electric dipole transitions for the hydrogen atom are governed by the matrix element

$$\langle n', l', m' | z | n, l, m \rangle$$

Interpret the $z$ operator as a tensor operator component, and derive conditions on the quantum numbers for this matrix element to be non-vanishing.

Problem 2 (Wigner-Eckart Theorem)

If $2l + 1$ operators $T_{lM}$, $M = -l, -l+1, \ldots, l$ satisfy the commutation relations (in units for which $\hbar = 1$)

$$[J_{\pm}, T_{lM}] = \sqrt{l(l + 1) - M(M \pm 1)} T_{l(M \pm 1)} \quad \text{and} \quad [J_3, T_{lM}] = MT_{lM}$$

then these are the components of an $SU(2)$ tensor operator of order $l$. In addition, one can define Cartesian components, e.g. for $l = 1$,

$$T_1 = \frac{1}{\sqrt{2}}(T_{1,-1} - T_{1,1}), \quad T_2 = \frac{i}{\sqrt{2}}(T_{1,1} + T_{1,-1}), \quad T_3 = T_{1,0}$$

(a) Show that the components of an $l = 1$ tensor operator satisfy

$$[J_a, T_b] = i \sum_{k=1}^{3} \epsilon_{abc} T_k$$

(1)

Generalize the definition of tensor operator to irreducible representations of a general compact Lie algebra (not necessarily $SU(2)$). Prove the analogue of eq. (1) in the general case.

The Wigner-Eckart Theorem states: matrix elements of $T_{lM}$ with eigenstates of angular momentum depend on the $m$ quantum numbers only via Clebsch-Gordan coefficients, where the factor $\langle j || T_i || j' \rangle$ is called the reduced matrix element and is independent of $m$ quantum numbers:

$$\langle j'm' | T_{lM} | jm \rangle = \langle jmlM | jl'j'm' \rangle \frac{1}{\sqrt{2j + 1}} \langle j'|| T_i || j \rangle$$

(b) Without using the theorem, show that the following condition for a non-vanishing matrix element holds:

$$m + M = m'$$

(c) For a scalar operator $S = T_{00}$ ($l = 0$), show that

$$\langle j'm' | S | jm \rangle \sim \delta_{jj'} \delta_{mm'}$$

and the normalization constant does not depend on $m$. 
(d) For a vector operator, show that a non-vanishing matrix element requires
\[ |\Delta j| = |j - j'| \leq 1 \quad \text{and} \quad |\Delta m| = |m - m'| \leq 1 \]
with the additional condition that \( j = j' = 0 \) is not allowed.
(e) Show that angular momentum is a vector operator, with appropriate choices of \( J_l \), where \( l = -1, 0, 1 \).
(f) For the components \( V_{M} \) of a vector operator, show the projection theorem:
\[
\langle j m' | V_{M} | j m \rangle = \frac{1}{j(j+1)} \langle j \tilde{m} | \vec{J} \cdot \vec{V} | j \tilde{m} \rangle \langle j m' | J_{M} | j m \rangle
\]
where \( \tilde{m} \) is arbitrary. What does this say about the action of \( \vec{J} \cdot \vec{V} \) on spaces of constant \( j \)?
(g) Placing a Hydrogen atom in an electric quadrupole field gives rise to the perturbation \( H' = cr^2 Y_{20} \). Show (using the Wigner-Eckart theorem) that all matrix elements \( \langle \psi | H' | \psi' \rangle \) between eigenstates \( \psi, \psi' \) in the first and second shell vanish, except \( \Delta E_m = \langle \psi_m | H' | \psi_m \rangle \), where \( \psi_m \) denotes a state with angular momentum quantum number \( \ell = 1 \) and magnetic quantum number \( m \) in the second main shell. In addition, show that \( \Delta E_1 = \Delta E_{-1} = -\frac{1}{2} \Delta E_0 \).

**Problem 3 (Roots and Weights)**


**Problem 4 (Spin-Statistics for Dirac Field)**

Assume that \( \psi \) is a Dirac spinor with the “wrong” spin-statistics connection,
\[
[\psi(x), \psi(y)] = 0 \quad \text{for} \quad (x - y)^2 < 0
\]
Define
\[
W(x, y) = \langle \Omega_0, \psi(x)\psi^*(y)\Omega_0 \rangle, \quad \hat{W}(y, x) = \langle \Omega_0, \psi^*(y)\psi(x)\Omega_0 \rangle
\]
(a) Prove using the transformation properties of the fields under the Poincaré group that \( W, \hat{W} \) depend only on the difference \( x - y \), so we can write
\[
W(x, y) = W(x - y), \quad \hat{W}(y, x) = \hat{W}(y - x)
\]
(b) Show using transformation properties of the fields under \( SL(2, \mathbb{C}) \) that
\[
W(\zeta) = \hat{W}(-\zeta) \quad \text{and} \quad \hat{W}(\zeta) = -\hat{W}(-\zeta) \quad (2)
\]
(c) Prove that equations (2) imply that \( \psi(x)\Omega_0 = 0 \).

*Hint:* show that
\[
||\psi(f^*)\Omega_0||^2 + ||\psi(\hat{f})\Omega_0||^2 = 0
\]
where \( \hat{f}(x) := f(-x) \) and \( \psi(f) = \int \psi(x)f(x)dx \).
Problem 5 (Euclidean Green’s functions)

(a) Using Fourier transform methods, calculate the Green’s function \( G_0^E(x - x') \) of the operator \(-\nabla^2 + m^2\) in real \( d\)-dimensional Euclidean space. Also show that this expression is equivalent to the expectation value of the time-ordered product derived in class, under analytic continuation to imaginary time, \( x_0 \to ix_0 \). \( \text{Answer: } G_0^E(x - x') = \int \frac{d^dp}{(2\pi)^d} \frac{\exp(ip(x_\mu - x'_\mu))}{p^2 + m^2} \)

(b) Compute the integral in part (a) in terms of Bessel functions, writing out the details of any change of variables or Gaussian integrations that you may need use. Discuss the long-distance and short-distance asymptotics of the resulting Green’s function.

(c) In any quantum field theory, the Heisenberg representation of the field operator \( \hat{\phi} \) in imaginary time (with \( \hbar = 1 \)) is

\[
\hat{\phi}(\vec{x}, \tau) = e^{iH\tau} \hat{\phi}(\vec{x}, 0) e^{-iH\tau}
\]

and the Green’s function or 2-point function is defined to be

\[
G^{(2)}(\vec{x} - \vec{x}', \tau - \tau') = \langle 0 | T \hat{\phi}(\vec{x}, \tau) \hat{\phi}(\vec{x}', \tau') | 0 \rangle
\]

The theory is said to have a mass gap if the lowest and first-excited eigenvalues of the Hamiltonian are separated by an interval of nonzero length. Prove that in any quantum field theory with a mass gap, the Green’s functions \( G^{(2)} \) decay exponentially in imaginary time, while in real time we get an oscillatory behavior.

Problem 6 (Bound States)

Consider the tensor product \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \) in which \( \mathcal{H}_i \) is isomorphic to the two-dimensional defining representation of \( SU(2) \) for \( i = 1, 2 \). This represents the quantum mechanical combination of two spin-\( \frac{3}{2} \) particles.

(a) Show that \( \mathcal{H} \) decomposes as a direct sum of two irreducible representations, and determine the total spin and the dimension of each. Also determine whether the states in each irreducible representation are symmetric or antisymmetric under the interchange of the two spins.

(b) Use the information from part (a) to compute the quantum numbers under \( P \) (parity) and \( C \) (charge conjugation) of all electron-positron bound states with \( S, P, \) or \( D \) wave functions.