Introduce the $2n$ independent generators of a Grassmann algebra, $\chi_k$, $\chi_k^*$, $k = 1, \ldots, n$. These generators all anticommute with each other:

$$\chi_k \chi_l = -\chi_l \chi_k, \quad \chi_k^* \chi_l^* = -\chi_l^* \chi_k^*, \quad k, l = 1, \ldots, n. \tag{1}$$

The variables $\chi_k$ and $\chi_k^*$ are independent. According to (1), the square of a Grassmannian variable is zero. As a consequence, any function of these variables, when expanded in a power series, may contain only the terms of the zeroth and the first order in each Grassmann variable. For this reason, integration over the Grassmann algebra can be defined uniquely by the following rules:

$$\int \chi_k d\chi_k = \int \chi_k^* d\chi_k^* = \frac{1}{\sqrt{2\pi}} \delta_{kl}, \quad \text{and} \quad \int d\chi_k = \int d\chi_k^* = \int \chi_k d\chi_k^* = \int \chi_k^* d\chi_k = 0. \tag{2}$$

Note the normalization that we use here, $\int \chi_k d\chi_k = \frac{1}{\sqrt{2\pi}}$, is different from the one in class $\int \chi_k d\chi_k = 1$ (the more normal one for Grassmann integration). The advantage of the normalization that we use here is that the Gaussian integral $\int d\Phi d\Phi^* \exp(-\Phi^* \Phi)$ requires no special normalization factor, where

$$d\Phi d\Phi^* = dS_1 dS_1^* \cdots dS_n dS_n^* d\chi_1 \cdots d\chi_n d\chi_n^*. \tag{3}$$

The differentials $d\chi_k$, $d\chi_k^*$ anticommute with each other and with the Grassmannian variables. It is convenient to define an antilinear conjugation for the Grassmann variables by the following rules

$$(\chi_k)^* = \chi_k^*, \quad (\chi_k^*)^* = \chi_k, \quad (\chi_k \chi_l)^* = \chi_k^* \chi_l^*. \tag{4}$$

One easy conceptual way to achieve relations (1), (2), and (4) for “complex” Grassmann generators is to take $2n$ independent generators of a Grassmann algebra $\xi_k$ and $\eta_k$ for $k = 1, \ldots, n$. Then define $\xi_k^* = \xi_k$ and $\eta_k^* = \eta_k$, $(\xi_k \xi_l)^* = \xi_k \xi_l$, etc., and extend this relation by anti-linearity to linear combinations of products of these generators. For these “real” variables,

$$\int \xi_k d\xi_l = \int \eta_k d\eta_l = \delta_{kl}, \quad \text{and} \quad \int d\xi_k = \int d\eta_k = \int \xi_k d\eta_l = \int \eta_k d\xi_l = 0. \tag{5}$$

Finally, define the complex generators of the algebra by

$$\chi_k = \frac{1}{\sqrt{2}} (\xi_k + i \eta_k), \tag{6}$$

and

$$d\chi_k = \frac{1}{\sqrt{2}} (d\xi_k - i d\eta_k). \tag{7}$$

(Note the minus sign!)

**Problem 0.**

Show that relations (1), (2), and (4) follow.
Define a supervector by
\[
\Phi = \begin{pmatrix}
S_1 \\
\vdots \\
S_n \\
\chi_1 \\
\vdots \\
\chi_n
\end{pmatrix},
\] with the adjoint \( \Phi^\dagger = (S_1^*, \ldots, S_n^*, \chi_1^*, \ldots, \chi_n^*) \), \( (8) \)

where \( S_i \) are complex numbers and \( \chi_i \) are defined above. A supermatrix has the structure
\[
K = \begin{pmatrix}
a & \sigma \\
\rho & b
\end{pmatrix},
\] \( (9) \)

where the boson-boson (bb) block \( a \) and the fermion-fermion (ff) block \( b \) are ordinary \( n \times n \) matrices, while the boson-fermion (bf) and fermion-boson (fb) blocks \( \sigma, \rho \) are \( n \times n \) matrices with anticommuting entries. For any two supervectors \( \Phi, \Psi \), the tensor product \( (\Phi \otimes \Psi^\dagger)_{ij} = \Phi_i (\Psi^\dagger)_j \) is a supermatrix. The supertrace and the superdeterminant of the supermatrix \( (9) \) are defined as follows
\[
\text{Str} K = \text{Tr} a - \text{Tr} b ,
\] \( (10) \)
\[
\text{Sdet} K = \exp \text{Str} \ln K
\] \( (11) \)

Finally, the hermitian conjugate of a supermatrix is defined by
\[
K^\dagger = \begin{pmatrix}
a^\dagger & \rho^\dagger \\
\sigma^\dagger & b^\dagger
\end{pmatrix},
\] \( (12) \)

where for the anticommuting (bf and fb) blocks the usual definition holds, \( \sigma^\dagger = (\sigma^*)^T \).

**Problem 1.**

Show that with the above set of definitions, the following properties of supervector and supermatrix algebra are valid:

(a) if \( K \) is a supermatrix and \( \Phi \) a supervector, then \( \Psi = K\Phi \) is a supervector;
(b) hermitian conjugation satisfies the usual properties
\[
K = \Phi \otimes \Psi^\dagger \Longrightarrow K^\dagger = \Psi^\dagger \otimes \Phi^\dagger ,
\] \( (13) \)
\[
(\Phi^\dagger K\Psi)^\dagger = \Psi^\dagger K^\dagger \Phi ,
\] \( (14) \)
\[
(K^\dagger)^\dagger = K
\] \( (15) \)

*Note:* One could use \( (14) \) as a definition of \( K^\dagger \) instead of the definition above.

(c) supertrace and superdeterminant of a product:
\[
\text{Str} K_1 K_2 = \text{Str} K_2 K_1 ,
\] \( (16) \)
\[
\text{Sdet} K_1 K_2 = \text{Sdet} K_1 \cdot \text{Sdet} K_2 .
\] \( (17) \)

(d) Verify the triangular decompositions
\[
K = \begin{pmatrix}
a - \sigma b^{-1} & \sigma b^{-1} \\
0 & I
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
\rho & b
\end{pmatrix}
= \begin{pmatrix}
I & 0 \\
\rho a^{-1} & b - \rho a^{-1} \sigma
\end{pmatrix}
\begin{pmatrix}
a & \sigma \\
0 & I
\end{pmatrix}
\]

where \( I \) denotes the \( n \times n \) identity matrix.
(e) Show that
\[ \exp \text{Str} \ln K = \det(a - \sigma b^{-1} \rho) \det^{-1} b = \det a / \det(b - \rho a^{-1} \sigma) \]
thus giving an explicit expression for the superdeterminant (11). *Hint:* use (d).

(f) Show that the standard Gaussian super integral requires no normalization,
\[ \int d\Phi^\dagger d\Phi \exp(-\Phi^\dagger K \Phi) = 1 \]  
(18)
Verify the super-Gaussian relations
\[ \int d\Phi^\dagger d\Phi \exp(-\Phi^\dagger K \Phi) = S\det^{-1}(19) \]
\[ \int d\Phi^\dagger d\Phi \Phi^\alpha \Phi^\dagger \beta \exp(-\Phi^\dagger K \Phi) = (K^{-1})^{\alpha \beta} S\det^{-1}. \]  
(20)

Assume the boson-boson block \( a \) of the matrix \( K \) has positive definite real part, \( \Re S^\dagger a S > 0 \) for all \( S \neq 0 \), so that the integral over the bosonic components \( S \) of the supervector \( \Phi \) converges.

**Problem 2.**
This problem deals with an ensemble of \( N \times N \) Hermitian matrices \( H = H^\dagger \) with probability density
\[ P(H) = N \exp \left\{ -\frac{N}{2} \text{Tr}(H^2) \right\} \]  
(21)
where \( N \) is a normalization factor, chosen so that \( \int dH P(H) = 1 \), and \( dH \) denotes an \( N^2 \)-dimensional integration over the components of the matrix \( H \). The matrices \( H \) that we consider in this problem are ordinary Hermitian matrices (not supermatrices) with matrix elements denoted \( H_{ij} \). Then
\[ dH = dH_{11} dH_{22} \cdots dH_{NN} \prod_{1 \leq i < j \leq N} dH_{ij} \]

Also, \( \int dH \) means that each diagonal (real) entry is integrated from \(-\infty \) to \( \infty \), while each (complex) entry \( H_{ij} \) above the diagonal is integrated over \( \mathbb{C} \). This automatically ensures that the entries below the diagonal are treated appropriately.

The expectation of a function \( f(H) \) is written
\[ \langle f \rangle = \int f(H) P(H) dH. \]  
(22)
The matrix elements of \( H \) thus have a Gaussian distribution with mean \( \langle H_{ij} \rangle = 0 \) and variance
\[ \langle H_{ij} H_{i'j'}^\dagger \rangle = \frac{1}{N} \delta_{ii'} \delta_{jj'} \]
This is called the *Gaussian Unitary Ensemble.*

(a) In what follows, \( \Phi_i \) denote \( N \) different supervectors with \( n = 1 \). Prove that
\[ \langle (E - H)_{kl}^{-1} \rangle = -i \int d\Phi^\dagger d\Phi S_k S_l^\dagger \exp \left( i \sum_{ij} (E \delta_{ij} - H_{ij}) \Phi^\dagger_i \Phi_j \right), \quad \text{where} \quad \Phi_i = \begin{pmatrix} S_i \\ \chi_i \end{pmatrix} \]  
(23)
and we have included an infinitesimally small imaginary part \(+i\epsilon\), where \( \epsilon > 0 \) in the definition of \( E \), so that \( \text{Im} E > 0 \). This means we take the limit of the integral as \( \epsilon \to 0 \).
(b) Show that the expectation of the terms involving $H$ in the previous formula is given by
\[
\langle \exp(i \sum_{ij} \Phi_i^\dagger H_{ij} \Phi_j) \rangle = \exp \left\{ -\frac{1}{2N} \sum_{ij} (\Phi_i^\dagger \Phi_j)(\Phi_j^\dagger \Phi_i) \right\} .
\] (24)

(c) Now we introduce supermatrices into the problem. Let $\int dR$ be the integral over the space of all $2 \times 2$ supermatrices $R$ of the form
\[
R = \begin{pmatrix} q_b & \rho^* \\ \rho & iq_f \end{pmatrix} ; \quad q_b, q_f \in \mathbb{R} .
\] (25)

In this case, $\int dR$ is an ordinary double integral over the real variables $q_b, q_f \in \mathbb{R}$, followed by Grassmann integration over the independent variables $\rho, \rho^*$. Prove that
\[
\exp \left\{ -\frac{1}{2N} \sum_{ij} (\Phi_i^\dagger \Phi_j)(\Phi_j^\dagger \Phi_i) \right\} = \int dR \exp \left\{ -\frac{N}{2} \text{Str} R^2 - i \sum_i \Phi_i^\dagger R \Phi_i \right\} .
\] (26)

thus decoupling the $\Phi^4$ term in Eq. (24). Also show that (25) ensures convergence of the Gaussian integral over the commuting components.

(d) Show that substituting (26), (24) in (23), we get
\[
\langle \text{Tr}(E - H)^{-1} \rangle = -i \int dR \int d\Phi^* d\Phi \sum_k S_k S_k^* \exp \left\{ iE \sum_i \Phi_i^\dagger \Phi_i - \frac{N}{2} \text{Str} R^2 - i \sum_i \Phi_i^\dagger R \Phi_i \right\} .
\] (27)

(e) Show that this expression simplifies to
\[
\langle \text{Tr}(E - H)^{-1} \rangle = N \int dR (E - R)^{-1} \text{det}(-NS[R]) ,
\] where
\[
S[R] = \frac{1}{2} \text{Str} R^2 + \text{Str} \ln(E - R) .
\] (28)

The function $S[R]$ plays the role of an action in physics, both here and in the next problem, and bears no relation to the complex variables $S_i$ introduced above.

Hint: The integral over $\Phi$, if taken first, is now Gaussian (and convergent due to $\text{Im}E > 0$). Furthermore, since the matrix $R$ does not have any structure in the $N$-dimensional space (where the matrices $H$ act), the integral over each of $\Phi_i$, $\Phi_i^\dagger (i = 1, 2, \ldots, N)$ produces the same factor $\text{det}(E - R)$.

In the next problem, we use this representation to derive the famous Wigner semicircle law for the distribution of the density of states of a Random matrix.

A saddle point of a multivariate function is a stationary point that is not an extremum. We want to investigate the saddle point of the “action” $S[R]$. Evaluating $S[R]$ in the neighborhood of the saddle point will give a good approximation to the integral (27). Stationary points of an action are found as solutions of the equation $\delta S = 0$, where $\delta S \equiv S[R + \delta R] - S[R]$.

Problem 3.

(a) Show that the saddle point equation for the action (29) has the form
\[
R = (E - R)^{-1} .
\] (30)
Also show that the diagonal solutions of (30) take the form

$$ R = \frac{E}{2} - i\sqrt{1 - E^2/4} \begin{pmatrix} s_b & 0 \\ 0 & s_f \end{pmatrix}; \quad s_b, s_f = \pm 1. \quad (31) $$

Remarks: These matrices are not elements of the domain of integration as specified in (25). This issue can be resolved by shifting the integration contours for $R_{bb} = q_b$ and $R_{ff} = iq_f$. Careful consideration of the integration contours leads to the conclusion that only the saddle points with $s_b = +1$ are relevant, since $q_b = E$ is a singularity, and $\text{Im} E > 0$. You may assume without further justification that only the $s_b = +1$ saddle point contributes.

The $s_f = -1$ saddle point gives a contribution which is suppressed by a factor of $1/N$ (demonstrate this for extra credit). It will therefore be neglected in what follows and we will assume that the leading ($N \gg 1$) contribution is given by the vicinity of the saddle point

$$ R_0 = \frac{E}{2} - i\sqrt{1 - E^2/4}. \quad (32) $$

This gives rise to the famous Wigner semicircle law.

(b) Show that the preexponential factor in (28) evaluated at the saddle point, gives

$$ (E - R_0)^{-1}_{bb} = (R_0)_{bb} = \frac{E}{2} - i\sqrt{1 - E^2/4}. \quad (33) $$

(c) Calculate the integral in the saddle-point approximation, using (32).

Hint: You will need to prove the following expressions for the action at the saddle-point and the quadratic form around it,

$$ S[R_0] = 0, \quad (34) $$

$$ \delta^2 S[R_0] = C(\delta q_b^2 + \delta q_f^2 + 2\rho^*\rho); \quad C = 2 - E\left(\frac{E}{2} - i\sqrt{1 - E^2/4}\right). \quad (35) $$

where the notation $\rho, q_b, q_f$ in (35) is defined in (25).

(d) Explain how these results imply the density of states

$$ \nu(E) = \begin{cases} \frac{1}{\pi} \sqrt{1 - E^2/4}, & |E| \leq 2 \\ 0, & |E| \geq 2 \end{cases}, \quad (36) $$

for the Gaussian unitary ensemble.

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1It is implied in (31) that the first term is a constant $(E/2)$ times unit (super-)matrix. Likewise, we omit the unit matrix symbol in other formulas; e.g., in the l.h.s. (r.h.s.) of (28) $E$ implicitly includes the $N \times N$ (resp. $2 \times 2$) unit matrix.