AM 121: Intro to Optimization Models and Methods

Lecture 3: Applications, Examples, Exercises.

Yiling Chen
SEAS

Lecture 3: Lesson plan

• The Post Office Problem
• The SailCo Problem
• The SAVE-IT Company
• A Simple AMPL Example
  (AMPL: A Modeling Language for Mathematical Programming)

• From problem to LP + a little AMPL
Post Office Problem

Union rules state that each full-time employee must work 5 consecutive days and then receive 2 days off.

Formulate an LP to minimize the number of full-time employees who must be hired.

<table>
<thead>
<tr>
<th>Day</th>
<th>Number of full-time employees required</th>
</tr>
</thead>
<tbody>
<tr>
<td>1=Monday</td>
<td>17</td>
</tr>
<tr>
<td>2=Tuesday</td>
<td>13</td>
</tr>
<tr>
<td>3=Wednesday</td>
<td>15</td>
</tr>
<tr>
<td>4=Thursday</td>
<td>19</td>
</tr>
<tr>
<td>5=Friday</td>
<td>14</td>
</tr>
<tr>
<td>6=Saturday</td>
<td>16</td>
</tr>
<tr>
<td>7=Sunday</td>
<td>11</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\text{min } z &= \sum x_i \\
\text{s.t. } &\sum x_i - x_2 - x_3 \geq 17 \\
&\sum x_i - x_3 - x_4 \geq 13 \\
&\sum x_i - x_4 - x_5 \geq 15 \\
&\sum x_i - x_5 - x_6 \geq 19 \\
&\sum x_i - x_6 - x_7 \geq 14 \\
&\sum x_i - x_7 - x_1 \geq 16 \\
&\sum x_i - x_1 - x_2 \geq 11 \\
&x_i \geq 0
\end{align*}
\]

Note: will need solution to be integral!
Variation 1: Forced overtime

- Post office can force employees to work a 6th day each week
- Pay $100/day, $130 for the overtime day.

- Formulate an LP to minimize cost of meeting weekly labor requirements

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<td>14</td>
</tr>
<tr>
<td>6=Saturday</td>
<td>16</td>
</tr>
<tr>
<td>7=Sunday</td>
<td>11</td>
</tr>
</tbody>
</table>

Note: will need solution to be integral!

\[
\begin{align*}
\text{min } z &= 500 \sum_i x_i + 130 \sum_i o_i \\
\text{s.t. } \sum_i x_i - x_2 - x_3 + o_3 &\geq 17 \\
\sum_i x_i - x_3 - x_4 + o_4 &\geq 13 \\
\sum_i x_i - x_4 - x_5 + o_5 &\geq 15 \\
\sum_i x_i - x_5 - x_6 + o_6 &\geq 19 \\
\sum_i x_i - x_6 - x_7 + o_7 &\geq 14 \\
\sum_i x_i - x_7 - x_1 + o_1 &\geq 16 \\
\sum_i x_i - x_1 - x_2 + o_2 &\geq 11 \\
\end{align*}
\]

Note: will need solution to be integral!

\[
\begin{align*}
o_i &\leq x_i, \quad \forall i \\
x_i, o_i &\geq 0, \quad \forall i
\end{align*}
\]
Variation 2: Maximizing Weekends!

- Post office has 25 full-time employees. Cannot hire or fire.

- Formulate an LP to schedule employees in order to maximize the number of weekend days off received by the employees

<table>
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<tr>
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<td>14</td>
</tr>
<tr>
<td>6=Saturday</td>
<td>16</td>
</tr>
<tr>
<td>7=Sunday</td>
<td>11</td>
</tr>
</tbody>
</table>

Note: will need solution to be integral!

\[
\begin{align*}
\text{max } z &= x_7 + 2x_1 + x_2 \\
\text{s.t. } & \sum_i x_i - x_2 - x_3 \geq 17 \\
& \sum_i x_i - x_3 - x_4 \geq 13 \\
& \sum_i x_i - x_4 - x_5 \geq 15 \\
& \sum_i x_i - x_5 - x_6 \geq 19 \\
& \sum_i x_i - x_6 - x_7 \geq 14 \\
& \sum_i x_i - x_7 - x_1 \geq 16 \\
& \sum_i x_i - x_1 - x_2 \geq 11 \\
& \sum_i x_i = 25 \\
& x_i \geq 0
\end{align*}
\]
Variation 3: Part-time employees

- Now allowed to hire part-time employees
- Full-time: 8 hours a day, 5 consecutive days (2 days off). Cost $15/hour.
- Part-time: 4 hours a day, 5 consecutive days (2 days off). Cost $10/hour.
- Part-time limited by union to 25% of weekly labor (in terms of working hours).

<table>
<thead>
<tr>
<th>Day</th>
<th>Number of full-time employees required</th>
</tr>
</thead>
<tbody>
<tr>
<td>1=Monday</td>
<td>8*17=136 hrs</td>
</tr>
<tr>
<td>2=Tuesday</td>
<td>8*13=104 hrs</td>
</tr>
<tr>
<td>3=Wednesday</td>
<td>8*15=120 hrs</td>
</tr>
<tr>
<td>4=Thursday</td>
<td>8*19=152 hrs</td>
</tr>
<tr>
<td>5=Friday</td>
<td>8*14=112 hrs</td>
</tr>
<tr>
<td>6=Saturday</td>
<td>8*16=128 hrs</td>
</tr>
<tr>
<td>7=Sunday</td>
<td>8*11=88 hrs</td>
</tr>
</tbody>
</table>

Formulate an LP to minimize the post office’s weekly labor costs.

\[
\begin{align*}
\min z &= 120 \sum_i x_i + 40 \sum_i p_i \\
\text{s.t.} \\
8(\sum_i x_i - x_2 - x_3) + 4(\sum_i p_i - p_2 - p_3) &\geq 136 \\
8(\sum_i x_i - x_3 - x_4) + 4(\sum_i p_i - p_3 - p_4) &\geq 104 \\
8(\sum_i x_i - x_4 - x_5) + 4(\sum_i p_i - p_4 - p_5) &\geq 120 \\
8(\sum_i x_i - x_5 - x_6) + 4(\sum_i p_i - p_5 - p_6) &\geq 152 \\
8(\sum_i x_i - x_6 - x_7) + 4(\sum_i p_i - p_6 - p_7) &\geq 112 \\
8(\sum_i x_i - x_7 - x_1) + 4(\sum_i p_i - p_7 - p_1) &\geq 128 \\
8(\sum_i x_i - x_1 - x_2) + 4(\sum_i p_i - p_1 - p_2) &\geq 88 \\
4 \sum_i p_i &\leq 0.25(4 \sum_i p_i + 8 \sum_i x_i) \\
x_i, p_i &\geq 0
\end{align*}
\]

Note: will need solution to be integral!
The SailCo Problem

SailCo must determine how many sailboats to produce in each of next quarters. Must meet demand in each Q. Boats made in a Q can be used to meet demand in same Q.

Formulate an LP to determine a production schedule to minimize the sum of the production and inventory costs.

<table>
<thead>
<tr>
<th></th>
<th>Q1</th>
<th>Q2</th>
<th>Q3</th>
<th>Q4</th>
</tr>
</thead>
<tbody>
<tr>
<td>demand</td>
<td>40</td>
<td>60</td>
<td>75</td>
<td>25</td>
</tr>
</tbody>
</table>

production cost
$400/boat, first 40 boats in a quarter
$450/boat for additional boats

inventory cost
$20/boat/quarter for boats on hand at end of a quarter (after production has occurred and demand satisfied)

initial inventory: 10 sailboats at start of Q1

\[
\begin{align*}
\min z &= 400 \sum_t x_t + 450 \sum_t y_t + 20 \sum_t h_t \\
\text{s.t. } &x_t \leq 40, \quad \forall t \\
&h_1 = 10 + x_1 + y_1 - 40 \\
&h_2 = h_1 + x_2 + y_2 - 60 \\
&h_3 = h_2 + x_3 + y_3 - 75 \\
&h_4 = h_3 + x_4 + y_4 - 25 \\
&h_t, y_t, x_t \geq 0
\end{align*}
\]

\(h_t\): represents # boats on hand at end of quarter
\(x_t\): number of boats made less than 40
\(y_4\): number of boats made above 40
**Variation 1: Better handling of the Rolling Horizon**

- Notice that the plan leaves you with zero boats on hand at the end of Q4.
- Modify the LP to ensure that will end planning horizon with 10 boats in inventory.

- In practice, because this is a “rolling horizon” problem we would only implement for Q1 (produce 40 boats).
- Now observe demand. Suppose $d_1=35$. Then Q2 begins with inventory of $10+40-35=15$
- Now solve LP for Q2—Q5 and starting with 15 boats. Suppose projected demand for Q5 is 36

<table>
<thead>
<tr>
<th></th>
<th>Q1</th>
<th>Q2</th>
<th>Q3</th>
<th>Q4</th>
<th>Q5</th>
</tr>
</thead>
<tbody>
<tr>
<td>forecast demand</td>
<td>40</td>
<td>60</td>
<td>75</td>
<td>25</td>
<td>36</td>
</tr>
<tr>
<td>actual demand</td>
<td>35</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

keep inventory of 10 at end of planning horizon.

**Formulation for start of Q2:**

$$\min z = 400 \sum_t x_t + 450 \sum_t y_t + 20 \sum_t h_t$$

s.t. $x_t \leq 40, \quad \forall t$

$$h_2 = 15 + x_2 + y_2 - 60$$

$$h_3 = h_2 + x_3 + y_3 - 75$$

$$h_4 = h_3 + x_4 + y_4 - 25$$

$$h_5 = h_4 + x_5 + y_5 - 36$$

$$h_5 \geq 10$$

$h_t, y_t, x_t \geq 0$
Variation 2: Production-smoothing costs

- [Require at least 10 boats on hand at the end]
- Suppose that there is a concern about making production “smooth” across periods.
- Increase in production: $400/boat (training)
- Decrease in production: $500/boat (severance pay, decreasing morale, etc.)
- Assume 50 boats made during the Q preceding Q 1

- New constraints: \( x_2 + y_2 - (x_1 + y_1) = c_2 = c_2^+ - c_2^- \) to capture change in production from period to period

\[
\min z = 400 \sum_t x_t + 450 \sum_t y_t + 20 \sum_t h_t + 400 \sum_t c_t^+ + 500 \sum_t c_t^-
\]

s.t. \( x_t \leq 40, \ \forall t \)
\[
\begin{align*}
    h_1 &= 10 + x_1 + y_1 - 40 \\
    h_2 &= h_1 + x_2 + y_2 - 60 \\
    h_3 &= h_2 + x_3 + y_3 - 75 \\
    h_4 &= h_3 + x_4 + y_4 - 25 \\
    h_4 &\geq 10 \\
    x_1 + y_1 - 50 &= c_1^+ - c_1^- \\
    x_2 + y_2 - (x_1 + y_1) &= c_2^+ - c_2^- \\
    x_3 + y_3 - (x_2 + y_2) &= c_3^+ - c_3^- \\
    x_4 + y_4 - (x_3 + y_3) &= c_4^+ - c_4^- \\
    h_t, y_t, x_t, c_t^+, c_t^- &\geq 0
\end{align*}
\]
• Solution: (40,40,40,40) + (15,15,15,15)
• Why will the optimal solution not have both $c_t^+ > 0$ and $c_t^- > 0$?
• E.g., suppose production 70 in period 2 and 60 in period 1. What assignments to $c_t^+$ and $c_t^-$ are feasible?

Variation 3: Allowing demands to be backlogged
• Suppose now that demands can be backlogged and met in future periods.
• Penalty of $100/boat for each quarter that demand is unmet.
• Must meet all demand by end of Q4.
• Idea: modify the formulation so that
  
  \[ h_2^+ - h_2^- = h_1^+ - h_1^- + x_1 = y_1 - d_1 \]

  represents the potential backlog (at end of period) through $h_2^-$. Drop requirement that inventory be non-negative (except for final period).
• Include penalty in objective, e.g. add term $+100h_2^-$
The SAVE-IT Company

- Operates a reclamation center. Collects 4 types solid materials and treats so they can be amalgamated into a saleable product of 3 different grades.
- Formulate an LP to determine amount of each grade and mix of materials for each grade, maximizing profit (sales – amalg. cost)

Have grants of $30,000 for complete treatment cost (can’t treat more than this) and at least half of each material must be collected and treated.

<table>
<thead>
<tr>
<th>Material</th>
<th>Availability (lbs per week)</th>
<th>Treatment cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3000</td>
<td>3 $/lb</td>
</tr>
<tr>
<td>2</td>
<td>2000</td>
<td>6 $/lb</td>
</tr>
<tr>
<td>3</td>
<td>4000</td>
<td>4 $/lb</td>
</tr>
<tr>
<td>4</td>
<td>1000</td>
<td>5 $/lb</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Grade</th>
<th>Spec</th>
<th>Amalg. cost</th>
<th>Sales price</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1 &lt;= 30% 2 &gt;= 40% 3 &lt;= 50% 4 =20%</td>
<td>3 $/lb</td>
<td>8.5 $/lb</td>
</tr>
<tr>
<td>B</td>
<td>1 &lt;= 50% 2 &gt;= 10% 4 = 10%</td>
<td>2.5 $/lb</td>
<td>7 $/lb</td>
</tr>
<tr>
<td>C</td>
<td>1 &lt;= 70%</td>
<td>2 $/lb</td>
<td>5.5 $/lb</td>
</tr>
</tbody>
</table>
Variation 1: Pay for treatment

• SAVE-IT recognizes that it might be able to make more profits and improve environment.

• Negotiates “matching funds” that will provide a 80% rebate on treatment cost above 30,000 per week.

• Formulate an LP that (a) uses all of 30,000, (b) treats at least half of every material, (c) may in addition treat more if this is profitable.

\[
\begin{align*}
\text{max } & \quad 5.5 \sum_j x_{Aj} + 4.5 \sum_j x_{Bj} + 3.5 \sum_j x_{Cj} \\
\text{s.t. } & \quad x_{A1} \leq 0.3 \sum_j x_{Aj} \\
& \quad x_{A2} \geq 0.4 \sum_j x_{Aj} \\
& \quad x_{A3} \leq 0.5 \sum_j x_{Aj} \\
& \quad x_{A4} = 0.2 \sum_j x_{Aj} \\
& \quad x_{B1} \leq 0.5 \sum_j x_{Bj} \\
& \quad x_{B2} \geq 0.1 \sum_j x_{Bj} \\
& \quad x_{B4} = 0.1 \sum_j x_{Bj} \\
& \quad x_{C1} \leq 0.7 \sum_j x_{Cj} \\
& \quad 1500 \leq x_{A1} + x_{B1} + x_{C1} \leq 3000 \\
& \quad 1000 \leq x_{A2} + x_{B2} + x_{C2} \leq 2000 \\
& \quad 2000 \leq x_{A3} + x_{B3} + x_{C3} \leq 4000 \\
& \quad 500 \leq x_{A4} + x_{B4} + x_{C4} \leq 1000 \\
& \quad 3(x_{A1} + x_{B1} + x_{C1}) + 6(x_{A2} + x_{B2} + x_{C2}) \\
& \quad + 4(x_{A3} + x_{B3} + x_{C3}) + 5(x_{A4} + x_{B4} + x_{C4}) \leq 30000 \\
& \quad x_{A1}, x_{A2}, \ldots, x_{C4} \geq 0
\end{align*}
\]
• introduce

\[ y = 3(X_{A1} + x_{B1} + x_{C1}) + 6(x_{A2} + x_{B2} + x_{C2}) + 4(X_{A3} + x_{B3} + x_{C3}) + 5(x_{A4} + x_{B4} + x_{C4}) - 30000 \]

• \( y \geq 0 \)

• Add \((-0.8y)\) to the objective

\[
\begin{align*}
\max \ 5.5 \sum_j x_{A_j} + 4.5 \sum_j x_{B_j} + 3.5 \sum_j x_{C_j} - 0.8y \\
\text{s.t.} \quad x_{A1} &\leq 0.3 \sum_j x_{A_j} \\
x_{A2} &\geq 0.4 \sum_j x_{A_j} \\
x_{A3} &\leq 0.5 \sum_j x_{A_j} \\
x_{A4} &= 0.2 \sum_j x_{A_j} \\
x_{B1} &\leq 0.5 \sum_j x_{B_j} \\
x_{B2} &\geq 0.1 \sum_j x_{B_j} \\
x_{B4} &= 0.1 \sum_j x_{B_j} \\
x_{C1} &\leq 0.7 \sum_j x_{C_j} \\
1500 &\leq x_{A1} + x_{B1} + x_{C1} \leq 3000 \\
1000 &\leq x_{A2} + x_{B2} + x_{C2} \leq 2000 \\
2000 &\leq x_{A3} + x_{B3} + x_{C3} \leq 4000 \\
500 &\leq x_{A4} + x_{B4} + x_{C4} \leq 1000
\end{align*}
\]

\[ y = 3(x_{A1} + x_{B1} + x_{C1}) + 6(x_{A2} + x_{B2} + x_{C2}) + 4(x_{A3} + x_{B3} + x_{C3}) + 5(x_{A4} + x_{B4} + x_{C4}) - 30000 \]

\[ y, x_{A1}, x_{A2}, \ldots, x_{C4} \geq 0 \]
An AMPL Example

<table>
<thead>
<tr>
<th></th>
<th>Product A</th>
<th>Product B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Profit Per Unit</td>
<td>5</td>
<td>8</td>
</tr>
<tr>
<td>Machine time</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Storage Space</td>
<td>5</td>
<td>9</td>
</tr>
</tbody>
</table>

Total machine time = 6; Total storage space = 45;

\[
\begin{align*}
\text{max} & \quad 5x_1 + 8x_2 \\
\text{s.t.} & \quad x_1 + x_2 \leq 6 \\
& \quad 5x_1 + 9x_2 \leq 45 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]

Save by AMPL

- Install AMPL according to instructions
- Start AMPL at command line
  ```ampl
  model example.mod;
  data example.dat;
  option solver cplexamp;
  solve;
  display X;
  X [*] :=
  A  2.25
  B  3.75
  ;
  ```