Lecture 11: The Laplace Transform

The Laplace transform is a method for solving linear, time-invariant differential equations. It essentially turns differential equations into algebraic equations in the complex variable $s$. We have spent a significant amount of time solving different types of differential equations, which typically involves the solution of homogeneous and particular components of the problem. The Laplace transform method that we will discuss here solves both of these components simultaneously, and provides a very general approach to solving any type of linear time-invariant ODE.

**Definition**

Let $f(t)$ be a time function such that $f(t) = 0$ for $t < 0$. Furthermore, let $f(t)$ satisfy the following bounding condition:

$$\int_0^\infty |f(t)e^{-\alpha t}|dt < \infty$$

for some $\alpha \in \mathbb{R}$, $0 < \alpha < \infty$. If $f(t)$ satisfies this condition, then the Laplace transform of $f(t)$ exists, and is written:

$$L\{f(t)\} = \int_0^\infty f(t)e^{-st}dt = F(s)$$

where $L$ is the Laplace operator, and $s$ is a complex variable ($s = \sigma + j\omega$).

**Examples**

Let’s look at a few examples to demonstrate how the Laplace transform is applied.

**The Unit Step.** Consider the unit step function, which we have encountered earlier in the course:

$$f(t) = \begin{cases} 1 & t > 0 \\ 0 & \text{else} \end{cases}$$

This function often has the notation $1(t)$, to indicate a unit step that begins at time $t$, as shown in Figure 1.
Figure 1: The Unit Step Function

The unit step function $1(t)$, which is unity for $t > 0$ and 0 else.

The Laplace of this function is:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty e^{-st}dt = -\frac{1}{s}e^{-st}\bigg|_0^\infty = 0 - \left(-\frac{1}{s}\right) = \frac{1}{s}$$

Exponential Function. Consider the following function:

$$f(t) = e^{-\alpha t} \quad t \geq 0$$

$$= 0 \quad else$$

where $\alpha \in \mathbb{R}$. The Laplace transform of this function is:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty e^{-\alpha t}e^{-st}dt = \int_0^\infty e^{-(s+\alpha)t}dt = -\frac{1}{s + \alpha}e^{-(s+\alpha)t}\bigg|_0^\infty = \frac{1}{s + \alpha}$$

Ramp Function. Consider the following ramp function:

$$f(t) = Ct \quad t \geq 0$$

$$= 0 \quad else$$
where the constant $C \in \mathbb{R}$. The ramp function is shown in Figure 2.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{ramp_function.png}
\caption{The Ramp Function}
\end{figure}

The ramp function, which is $Ct$ for $t \geq 0$ and 0 else.

The Laplace transform of this function is:

$$
\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}dt
$$

$$
= \int_0^\infty Ct e^{-st}dt
$$

This can be done through integration by parts. Let:

$$
u = Ct$$

$$du = Cdt$$

and:

$$
\nu = \frac{-1}{s}e^{-st}
$$

$$dv = e^{-st}dt$$

So we have:

$$
F(s) = \int_0^\infty Ct e^{-st}dt
$$

$$
= uv\bigg|_0^\infty - \int_0^\infty vdu
$$

$$
= \frac{-Ct}{s} e^{-st}\bigg|_0^\infty - \int_0^\infty \frac{-1}{s}e^{-st}Cdt
$$

$$
= 0 + \frac{C}{s}\left(-\frac{1}{s}\right) e^{-st}\bigg|_0^\infty
$$

$$
= \frac{C}{s^2}
$$

These transforms, and many others, are in the Laplace transform tables handed out in class. The table provides you with a variety of transform pairs that are commonly used, although it is a good idea to understand how the transformation is done.
You might be wondering at this point how we can map the function $F(s)$ back to $f(t)$. The tables provide you with all of the necessary relationships between the two domains, but if that is unsatisfying to you, the expression for the inverse Laplace transform is:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st} ds$$

However, you won’t be required to use this for this course.

**Properties of the Laplace Transform**

There are a few properties of Laplace transforms that you will find very useful in analyzing a number of different types of systems.

**Multiplication by a Constant.** Since the Laplace transform is an integration, we might expect that the map is a linear one, and indeed it is. So it is very straightforward to show that:

$$\mathcal{L}\{kf(t)\} = k\mathcal{L}\{f(t)\} = kF(s)$$

where the constant $k \in \mathbb{R}$.

**Sums.** This is another property that you might already have a feel for, but just to be explicit, consider the following:

$$\mathcal{L}\{f_1(t) + f_2(t)\} = \int_{0}^{\infty} \left[f_1(t) + f_2(t)\right]e^{-st} dt$$

$$= \int_{0}^{\infty} f_1(t)e^{-st} dt + \int_{0}^{\infty} f_2(t)e^{-st} dt$$

$$= F_1(s) + F_2(s)$$

This is obviously the same for subtraction, and you also can imagine the combination of this property with the multiplication by constants.

**Translation.** Suppose that we have a function $f(t)$, that is multiplied by the unit step at time 0, so the function is 0 for $t < 0$. So we have $f(t)1(t)$. Suppose that we delay this function by time $\alpha \geq 0$, so we have $f(t - \alpha)1(t - \alpha)$, as shown in Figure 3.

The Laplace transform is:

$$\mathcal{L}\{f(t - \alpha)1(t - \alpha)\} = \int_{0}^{\infty} f(t - \alpha)1(t - \alpha)e^{-st} dt$$

Let $\tau = t - \alpha$. We then have:

$$\int_{0}^{\infty} f(t - \alpha)1(t - \alpha)e^{-st} dt = \int_{-\infty}^{0} f(\tau)1(\tau)e^{-s(\tau+\alpha)} d\tau$$

$$= \int_{0}^{\infty} f(\tau)1(\tau)e^{-s(\tau+\alpha)} d\tau$$

$$= e^{-\alpha s} F(s)$$
More Examples

Now that we know some of the properties of the Laplace transform, we can do more complicated things.

**Pulse Function.** Consider the pulse function, which is essentially when something is turned on, and then turned off after a delay. Explicitly we can write:

\[
f(t) = \begin{cases} 
  \frac{C}{t_0} & 0 < t < t_0 \\
  0 & \text{else} 
\end{cases}
\]

where the constant \( C \in \mathbb{R} \). We can think of this as the combination of two functions. The pulse function defined here is the standard step function \( 1(t) \) with height \( \frac{C}{t_0} \), minus the same function delayed by time \( t_0 \):

\[
f(t) = \frac{C}{t_0} 1(t) - \frac{C}{t_0} 1(t - t_0)
\]

This construct is shown in Figure 4.

Figure 4: The Pulse Function

\[
\begin{array}{c}
\text{A pulse function, with height } \frac{C}{t_0} \text{ and width } t_0. \text{ Note that this integrates to } C.
\end{array}
\]
The Laplace transform of this function is:

\[ \mathcal{L}\{f(t)\} = \frac{C}{t_0 s} - \frac{C}{t_0 s} e^{s t_0} \]
\[ = \frac{C}{t_0 s} \left(1 - e^{-s t_0}\right) \]

**Impulse Function.** Consider now a special case of the pulse function, the impulse function. We have dealt with the unit impulse function before, and typically denote it \( \delta(t) \). This is the Dirac delta function. Suppose that we have the following:

\[ f(t) = \lim_{t_0 \to 0} \frac{C}{t_0} \begin{cases} 0 < t < t_0 \\ 0 \text{ else} \end{cases} \]

This function is shown in Figure 5.

![Figure 5: The Impulse Function](image)

A Dirac delta function. Note that this has infinite height, infinitesimal width, and integrates to 1.

The Laplace of this function is:

\[ \mathcal{L}\{f(t)\} = \lim_{t_0 \to 0} \frac{C}{t_0 s} \left(1 - e^{-s t_0}\right) \]
\[ = \lim_{t_0 \to 0} \frac{d}{dt_0} \left(C(1 - e^{-s t_0})\right) \]
\[ = \frac{C s}{s} = C \]

**Differentiation.** With differential equations, we commonly deal with first, second, and even higher order derivatives. So it is useful to understand what this means in terms of the Laplace transform. We can show that:

\[ \mathcal{L}\left\{ \frac{d f(t)}{dt} \right\} = s F(s) - f(0) \]
where:

\[ f(0) = \lim_{t \to 0} f(t) \]

or in other words, \( f(0) \) is the initial condition of \( f(t) \). In general, for \( n \)th order derivatives, we have:

\[
\mathcal{L} \left\{ \frac{d^n f(t)}{dt^n} \right\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f(1)(0) - \cdots - f^{(n-1)}(0)
\]

As an example, consider the spring-mass-damper system \( m \ddot{y} + b \dot{y} + ky \):

\[
\mathcal{L} \{ m \ddot{y} + b \dot{y} + ky \} = m [s^2 Y(s) - s \dot{y}(0) - \dot{y}(0)] + b [s Y(s) - y(0)] + k Y(s)
\]

where \( y(0) \) is the initial displacement, and \( \dot{y}(0) \) is the initial velocity.

**Integration.** For integration, we have:

\[
\mathcal{L} \left\{ \int_0^t f(\tau) d\tau \right\} = \frac{F(s)}{s} + \frac{f^{-1}(0)}{s}
\]

where:

\[
f^{-1}(0) = \int_0^t f(\tau) d\tau \bigg|_{t=0}
\]

Typically this condition is zero, so for the general case we have:

\[
\mathcal{L} \left\{ \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_n} f(\tau) d\tau dt_1 dt_2 \cdots dt_{n-1} \right\} = \frac{F(s)}{s^n}
\]

**The Final Value Theorem (FVT).** One use of the Laplace transform is in the determination of the limit of the function as time goes to infinity. We have:

\[
\lim_{t \to \infty} f(t) = \lim_{s \to 0} s F(s)
\]

This applies if the limit on the left exists, which occurs if all of the roots of the denominator of \( s F(s) \) have negative real parts.

As an example, consider the following:

\[
F(s) = \frac{5}{s(s^2 + s + 2)}
\]

The FVT only applies if \( s F(s) \) satisfies the above condition. Does it?

\[
s F(s) = \frac{5s}{s(s^2 + s + 2)} = \frac{5}{s^2 + s + 2}
\]
and the roots of the denominator have negative real parts. So we can write:

\[ \lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s) = \lim_{s \to 0} \frac{5}{s^2 + s + 2} = \frac{5}{2} \]

But now consider a second example. What if:

\[ F(s) = \frac{\omega}{s^2 + \omega^2} \]

From this, we know that the roots are \( \pm j\omega \). Since the roots do not have negative real parts (they are zero in this case), we cannot apply the FVT.

**The Initial Value Theorem (IVT).** Similarly, there is an initial value theorem that can be used to determine the initial value of the function, from the Laplace transform:

\[ \lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s) \]

This theorem applies if the limit on the right exists, and if \( f(t) \) and \( \frac{df(t)}{dt} \) are both Laplace transformable.

**Convolution.** One mathematical function that we have not used yet, but will introduce in the near future is convolution. The convolution between two functions \( f_1(t) \) and \( f_2(t) \) has the form:

\[ f_1(t) * f_2(t) = \int_0^t f_1(\tau)f(t - \tau)d\tau \]

This is like flipping \( f_2(t) \) in time, and sliding it over \( f_1(t) \). It can be shown that the Laplace transform is:

\[ \mathcal{L}\{f_1(t) * f_2(t)\} = F_1(s)F_2(s) \]

The take-home message here is that convolution in the time-domain is equivalent to multiplication in the Laplace domain. That simple fact will turn out to be a really handy relationship.

**Partial Fraction Expansion**

We now know how to directly compute the Laplace transform of functions in time. However, once we have some general understanding of the Laplace transform, it is helpful to utilize the tables that have been provided of common transform pairs. The pairs in the typical transform table are obviously not the only functions you might encounter. However, what we will see here is that you will be able to express the functions you encounter as a linear combination of the very fundamental Laplace transforms in the table.
Suppose we have $Y(s)$ that is a rational function of $s$:

$$Y(S) = \frac{Q(s)}{P(s)}$$

where $Q(s)$ and $P(s)$ are polynomials of $s$. We assume that the order of $P(s)$ is greater than the order of $Q(s)$, which is a point we will come back to later in the course. So we have:

$$P(s) = s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1s + a_0$$

We want:

$$y(t) = L^{-1}\left\{\frac{Q(s)}{s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0}\right\}$$

**Distinct Roots**

Suppose we can factor the denominator into $n$ distinct roots:

$$Y(s) = \frac{Q(s)}{(s + p_1)(s + p_2)\cdots(s + p_n)}$$

where $p_1 \neq p_2 \neq \cdots \neq p_n$. So we don’t know how to take the Laplace transform of this $n^{th}$ order polynomial, but we do know how to take the transform of a linear combination of first order terms. Suppose that we can write:

$$Y(s) = \frac{K_1}{s + p_1} + \frac{K_2}{s + p_2} + \cdots + \frac{K_n}{s + p_n}$$

How do we solve for the constants $K_1, K_2, \ldots, K_n$?

$$K_i = \left. [(s + p_i)Y(s)] \right|_{s = -p_i}$$

**Example.** To nail this down, let’s do an example. Consider the following:

$$Y(s) = \frac{5s + 3}{(s + 1)(s + 2)(s + 3)}$$

$$= \frac{k_1}{s + 1} + \frac{k_2}{s + 2} + \frac{k_3}{s + 3}$$

Using the above technique, we have:

$$K_1 = \left. (s + 1)\frac{5s + 3}{(s + 1)(s + 2)(s + 3)} \right|_{s = -1}$$

$$= -5 + 3 \frac{1}{(1)} = -1$$

$$K_2 = \left. (s + 2)\frac{5s + 3}{(s + 1)(s + 2)(s + 3)} \right|_{s = -2}$$

$$= -10 + 3 \frac{1}{(-1)(1)} = 7$$
\[ K_3 = \left. \frac{5s + 3}{(s + 1)(s + 2)(s + 3)} \right|_{s=-3} \]
\[ = \frac{-15 + 3}{(2)(-1)} = -6 \]

So we have:
\[ Y(s) = \frac{-1}{s + 1} + \frac{7}{s + 2} - \frac{6}{s + 3} \]

and the corresponding transform is:
\[ y(t) = L^{-1} \left\{ \frac{-1}{s + 1} \right\} + L^{-1} \left\{ \frac{7}{s + 2} \right\} + L^{-1} \left\{ \frac{-6}{s + 3} \right\} \]
\[ = -e^{-t} + 7e^{-2t} - 6e^{-3t} \]

**Repeated Roots**

Now suppose that instead of distinct roots, we have some repeated roots:

\[ Y(s) = \frac{Q(s)}{(s + p_1)(s + p_2) \ldots (s + p_n - r)(s + p_n - r + 1)^r} \]

where there \( n - r \) distinct roots \((p_1 \neq p_2 \neq \ldots \neq p_n - r)\), and a root repeated \( r \) times. We can write \( Y(s) \) as the following:

\[ Y(s) = \frac{K_1}{s + p_1} + \frac{K_2}{s + p_2} + \ldots + \frac{K_{n-r}}{s + p_{n-r}} \]
\[ + \frac{A_1}{(s + p_{n-r+1})} + \frac{A_2}{(s + p_{n-r+1})^2} + \ldots + \frac{A_r}{(s + p_{n-r+1})^r} \]

where the \( K_1, K_2, \ldots K_{n-r} \) can be obtained as before. For \( A_1, A_2, \ldots A_r \), we have:

\[ A_r = \left. (s + p_{n-r+1})^r Y(s) \right|_{s=-p_{n-r+1}} \]
\[ A_{r-1} = \frac{d}{ds} \left. (s + p_{n-r+1})^r Y(s) \right|_{s=-p_{n-r+1}} \]
\[ A_{r-2} = \frac{1}{2!} \frac{d^2}{ds^2} \left. (s + p_{n-r+1})^r Y(s) \right|_{s=-p_{n-r+1}} \]
\[ \vdots \]
\[ A_1 = \frac{1}{(r-1)!} \frac{d^{r-1}}{ds^{r-1}} \left. (s + p_{n-r+1})^r Y(s) \right|_{s=-p_{n-r+1}} \]
Example. To nail this down, consider the following example:

\[
Y(s) = \frac{1}{s(s+1)^3(s+2)}
\]

\[
= \frac{K_1}{s} + \frac{K_2}{s+2} + \frac{A_1}{s+1} + \frac{A_2}{(s+1)^2} + \frac{A_3}{(s+1)^3}
\]

We can determine the constants in the following manner:

\[
K_1 = \left. [sY(s)] \right|_{s=0} = \frac{1}{1(1)(2)} = \frac{1}{2}
\]

\[
K_2 = \left. [(s+2)Y(s)] \right|_{s=-2} = \frac{1}{(-2)(-1)^3} = \frac{1}{2}
\]

\[
A_1 = \left. [(s+1)^3Y(s)] \right|_{s=-1} = \frac{1}{-4} = -1
\]

\[
A_2 = \left. \frac{d}{ds}[(s+1)^3Y(s)] \right|_{s=-1} = \left. \frac{d}{ds} \left[ \frac{1}{s(s+2)} \right] \right|_{s=-1} = 0
\]

\[
A_3 = \left. \frac{1}{2!} \frac{d^2}{ds^2}[(s+1)^3Y(s)] \right|_{s=-1} = \left. \frac{1}{2!} \frac{d^2}{ds^2} \left[ \frac{1}{s(s+2)} \right] \right|_{s=-1} = -1
\]

So we have:

\[
Y(s) = \frac{1}{2s} + \frac{1}{2(s+2)} - \frac{1}{s+1} - \frac{1}{(s+1)^3}
\]

Taking the inverse Laplace is just a matter of using the tables:

\[
y(t) = L^{-1} \{F(s)\}
\]

\[
= \frac{1}{2} 1(t) + \frac{1}{2} e^{-2t} - e^{-t} - \frac{1}{2} L^{-1} \left\{ \frac{2}{(s+1)^3} \right\}
\]

\[
= \frac{1}{2} 1(t) + \frac{1}{2} e^{-2t} - e^{-t} - \frac{1}{2} t^2 e^{-t}
\]

Using the Laplace Transform to Solve ODE’s

So far we have only just discussed the mathematics behind the Laplace transform and have given some examples and properties. Our original motivation was for solving ordinary differential equations. We now have all the tools to do just that. Consider the following example:

\[
\dot{y} + 2y = 0
\]
and let the initial displacement be $y(0) = 2$. Suppose that we take the Laplace transform of this equation. We do this by taking the Laplace of both sides:

$$L\{\dot{y} + 2y\} = sY(s) - y(0) + 2Y(s) = 0$$

Solving for $Y(s)$, we have:

$$Y(s) = \frac{y(0)}{s + 2} = \frac{2}{s + 2}$$

Taking the inverse Laplace, we find:

$$y(t) = L^{-1}\left\{\frac{2}{s + 2}\right\} = 2e^{-2t}$$

So that’s it. We have solved the differential equation in a really straightforward way.

What would have happened if the input were a unit impulse? We would then have:

$$L\{\dot{y} + 2y\} = sY(s) - y(0) + 2Y(s) = L\{\delta(t)\} = 1$$

giving us:

$$Y(s) = \frac{1 + y(0)}{s + 2} = \frac{3}{s + 2}$$

resulting in:

$$y(t) = L^{-1}\left\{\frac{3}{s + 2}\right\} = 3e^{-2t}$$

This is the same result, with a slightly larger magnitude. You can imagine doing this for a variety of different types of inputs, which is where the PFE comes in.

What if the input is a unit step? We then get the following:

$$L\{\dot{y} + 2y\} = sY(s) - y(0) + 2Y(s) = L\{1\} = \frac{1}{s}$$

So we get:

$$Y(s) = \frac{1 + y(0)}{s + 2} = \frac{2s + 1}{s(s + 2)}$$

Using our PFE technique, we can write:

$$Y(s) = \frac{K_1}{s} + \frac{K_2}{s^2}$$

Solving for $K_1$ and $K_2$, we get:

$$K_1 = \left[\frac{2s + 1}{s(s + 2)}\right]_{s=0} = \frac{1}{2}$$
\[ K_2 = \left. \left( s + 2 \right) \frac{2s + 1}{s(s + 2)} \right|_{s = -2} = \frac{3}{2} \]

So we get:

\[ y(t) = \mathcal{L}^{-1}\left\{ \frac{1}{2s} + \frac{3}{2(s + 2)} \right\} \]

\[ = \frac{1}{2} 1(t) + \frac{3}{2} e^{-2t} \]

That’s the basic idea behind using the Laplace transform to solve time-invariant ODE’s. Obviously it gets uglier when the inputs and/or the differential equation is complex, but you now have the tools to attack these problems.

Required reading: Khoo, Section 2.6 [1]

References