Response to a pure sinusoid

We previously stated that if an input to a linear system is sinusoidal \( u(t) = A \sin(\omega_0 t) \), then the output would be a sine wave at the same frequency shifted in phase and changed in magnitude. If we have:

\[
\frac{Y(s)}{U(s)} = G(s)
\]

then the output is:

\[
y(t) = |G(j\omega)| A \sin(\omega_0 t + \phi)
\]

where:

\[
|G(j\omega)| = \sqrt{[\text{Re}(G(j\omega))]^2 + [\text{Im}(G(j\omega))]^2}
\]

and:

\[
\phi = -\tan^{-1}\left(\frac{\text{Im}(G(j\omega))}{\text{Re}(G(j\omega))}\right)
\]

Note that the magnitude and phase shift depend on the frequency at which you are driving the system.

Example. Consider the following system:

\[
G(s) = \frac{10}{s^2 + 2s + 1}
\]

with an input \( u(t) = 5 \sin t \). We then can write:

\[
y(t) = |G(j)| 5 \sin(t + \phi)
\]

The transfer function \( G(s) \) evaluated at \( s = j\omega = j \) is:

\[
G(j) = \frac{10}{j^2 + 2j + 1} = \frac{10}{-1 + 2j - 1} = \frac{10}{2j} \cdot \frac{-2j}{-2j} = -5j
\]
To find the magnitude, we can write:

\[ |G(j)| = \sqrt{0^2 + 5^2} = 5 \]

The angle of a negative purely real number is \(-90^\circ\) or \(-\frac{\pi}{2}\) rad/sec. We can easily see this in the complex plane, as shown in Figure 1. The solution then takes the form:

\[ y(t) = 5 \cdot 5 \sin \left( t - \frac{\pi}{2} \right) = 25 \sin \left( t - \frac{\pi}{2} \right) \]

It is easy to see the effect of the system on this pure sinusoidal input graphically, as shown in Figure 2.

Note that if \( u(t) = 5 \sin t \), then \( \omega_0 = 1 \) rad/sec, so the frequency in cycles/sec is \( f_0 = \omega_0 / 2\pi = 1 / 2\pi \) cycles/sec. This gives a period of \( T_0 = 2\pi \approx 6 \) sec.

**The General Case**

In general, the above discussion is useful not because we typically experience purely sinusoidal inputs in real life. The utility lies in the linearity of the relationship. From Fourier analysis, we
know that any signal can be written as a linear combination of sine waves at different frequencies, magnitudes, and phases:

\[ u(t) = A_0 \sin(\omega_0 t + \psi_0) + A_1 \sin(\omega_1 t + \psi_1) + \ldots \]

Because the system is linear, the output is equal to the sum of the outputs as if you had applied each one of these sine waves individually. The output of the system to an input of this type is:

\[ y(t) = |G(j\omega)| \sin(\omega_0 t + \psi_0 + \phi_0) + |G(j\omega_1)| \sin(\omega_1 t + \psi_1 + \phi_1) + \ldots \]

So by knowing \(|G(j\omega)|\) and \(\angle G(j\omega) \forall \omega\), we can completely describe the output to any arbitrary input. For any transfer function, we can do this by simply evaluating the magnitude and phase angle over the entire frequency range. It’s a very straightforward thing to do using a computer, but we will find out that some very simple tools are pretty useful in sketching these relationships out by hand and will give you some insight as to the dynamics of the system. The type of plot that we will be generating is called a Bode plot, which will now be described.

**The Bode Plot**

The Bode plot is a plot of the magnitude and phase of a transfer function, as a function of frequency. We can write any transfer function in the following form:

\[
G(s) = \frac{N(s)}{D(s)} = \frac{K(s + z_1)(s + z_2)\ldots(s + z_m)}{(s + p_1)(s + p_2)\ldots(s + p_n)} e^{-\tau_d s}
\]

where each of the \(G_i(s)\) is of a very simple form. Taking a look at the magnitude, we have:

\[
|G(j\omega)| = |G_1(j\omega)G_2(j\omega)\ldots G_N(j\omega)|
\]

\[
= |G_1(j\omega)||G_2(j\omega)|\ldots|G_N(j\omega)|
\]

We often look at the log of the magnitude in decibels, which is defined as:

\[
|G(j\omega)|_{dB} = 20 \log_{10} |G(j\omega)| = 20[\log_{10} |G_1(j\omega)| + \log_{10} |G_2(j\omega)| + \ldots + \log_{10} |G_N(j\omega)|]
\]

\[
= 20 \log_{10} |G_1(j\omega)| + 20 \log_{10} |G_2(j\omega)| + \ldots + 20 \log_{10} |G_N(j\omega)|
\]

\[
= |G_1(j\omega)|_{dB} + |G_2(j\omega)|_{dB} + \ldots + |G_N(j\omega)|_{dB}
\]

So the utility is that the product turned into a sum. We’ll see why this is handy shortly.

Similarly, with the phase we have:

\[
\angle G(j\omega) = \angle G_1(j\omega) + \angle G_2(j\omega) + \ldots + \angle G_N(j\omega)
\]

So with both the magnitude and the phase angle, the different components sum. We can therefore look at each component, draw the magnitude and phase as a function of the frequency \(\omega\), and add.
**Example.** Consider the following system:

\[
G(s) = \frac{(s + z_1)(s + z_2)}{s(s + p_1)(s^2 + 2\zeta_\omega_n s + \omega_n^2)}e^{-T_d s}
\]

We can easily rearrange this:

\[
G(s) = \frac{K(1 + \frac{1}{z_1})(1 + \frac{1}{z_2})}{s(1 + \frac{1}{p_1})(1 + \frac{1}{\omega_n s + \frac{1}{\omega_n^2}})}e^{-T_d s}
\]

where \(K, z_1, z_2, p_1, \zeta, \omega_n, T_d \in \mathbb{R}\). The magnitude is:

\[
|G(j\omega)|_{dB} = 20\log_{10}|G(j\omega)|
\]

\[
= 20\log_{10}|K| + 20\log_{10}\left|1 + \frac{j\omega}{z_1}\right| + 20\log_{10}\left|1 + \frac{j\omega}{z_2}\right| - 20\log_{10}|j\omega|
\]

\[
-20\log_{10}\left|1 + \frac{j\omega}{p_1}\right| - 20\log_{10}\left|1 + \frac{2\zeta j\omega}{\omega_n} + \frac{(j\omega)^2}{\omega_n^2}\right| + 0
\]

The phase angle can be computed as:

\[
\angle G(j\omega) = \angle K + \angle\left(1 + \frac{j\omega}{z_1}\right) + \angle\left(1 + \frac{j\omega}{z_2}\right) - \angle(j\omega) - \angle\left(1 + \frac{j\omega}{p_1}\right) - \\
\angle\left(1 + \frac{2\zeta j\omega}{\omega_n} + \frac{(j\omega)^2}{\omega_n^2}\right) - \omega T_d
\]

**Bode Plot Elements**

So in general, we can break things down into these simple types of components no matter how complicated the transfer function is.

The types of elements break down into the following:

- **Constant gain** \(K \in \mathbb{R}\)
- **Poles/zeros at the origin**: \(G(s) = s\) or \(G(s) = \frac{1}{s}\)
- **Poles/zeros real at \(p_i\) or \(z_i\)**: \(G(s) = 1 + \frac{1}{s z_i}\) or \(G(s) = \frac{1}{1 + \frac{1}{p_i}}\)
- **Complex conjugate pair of poles/zeros**: \(G(s) = 1 + \frac{2\zeta s}{\omega_n} + \frac{s^2}{\omega_n^2}\) or \(G(s) = \frac{1}{1 + \frac{2\zeta s}{\omega_n} + \frac{s^2}{\omega_n^2}}\)
- **Pure time delay**: \(G(s) = e^{-T_d s}\)

Let’s look at the properties of each type.
Constant $K$. Suppose we have a constant $K \in \mathbb{R}$. The magnitude can be written:

$$|K|_{dB} = 20 \log_{10} |K| = \text{constant} \quad \forall \omega$$

The phase angle depends on the sign of $K$:

$$\angle K = \begin{cases} 0 & \text{if } k > 0 \\ -180^\circ & \text{else} \end{cases}$$

As an example, consider $G(s) = -1$. Let the input $u(t) = A \sin(\omega_0 t)$. The output can then be written as:

$$y(t) = |G(j\omega_0)| A \sin(\omega_0 t + \theta) = A \sin(\omega_0 t - \pi)$$

So the input and the output are completely out of phase with each other. Figure 3 shows the Bode plot for a constant gain of $K = 10$.

**Figure 3: Constant Gain Element**

Pole/Zero at the Origin. Suppose that we have a pole at the origin:

$$G(s) = \frac{1}{s}$$

The magnitude is:

$$|G(j\omega)|_{dB} = 20 \log_{10} \left| \frac{1}{j\omega} \right| = -20 \log_{10} |j\omega| = -20 \log_{10} \omega$$

This means that $|G(j\omega)|_{dB}$ vs. $\log_{10} \omega$ is a straight line with a slope of $-20$ dB/decade, which is often called db/decade. The magnitude is 0 dB at frequency $\omega = 1$. The phase angle is just the angle associated with $-j\omega$, which is $-90^\circ$:

$$\angle G(j\omega) = -90^\circ$$

The same argument can be made for a zero at the origin $G(s) = s$, except that the magnitude will be $|G(j\omega)|_{dB} = +20 \log_{10} \omega$, having a positive slope of 20 dB/decade and passing through 0 dB at $\omega = 1$. Figure 4 shows the Bode plot for a pole and a zero at the origin.
Real Pole/Zero. Suppose we have a zero at $z_i \in \mathbb{R}$:

$$G(s) = 1 + \frac{s}{z_i}$$

Evaluating at $s = j\omega$ gives us:

$$G(j\omega) = 1 + \frac{j\omega}{z_i}$$

The magnitude is:

$$|G(j\omega)|_{dB} = 20 \log_{10} |G(j\omega)| = 20 \log_{10} \sqrt{1^2 + \left(\frac{\omega}{z_i}\right)^2}$$

Of course we could plot this on the computer very easily, but to get a handle on it first, let’s look at the asymptotic properties. For small $\omega$ ($\omega/z_i \ll 1$) we have:

$$|G(j\omega)|_{dB} \approx 20 \log_{10} \sqrt{1^2 + 0^2} = 0dB$$

For large $\omega$ ($\omega/z_i \gg 1$) we have:

$$|G(j\omega)|_{dB} \approx 20 \log_{10} \sqrt{\frac{\omega^2}{z_i^2}} = 20 \log_{10} \frac{\omega}{z_i}$$

$$= 20 \log_{10} \omega - 20 \log_{10} z_i$$

This is a straight line, with slope $+20$dB/decade, and is 0dB at $\omega = z_i$. The straight-line approximation starts with an initial magnitude of 0dB, constant until $\omega = z_i$, at which point it breaks upward at $+20$dB/decade.

The phase angle can be expressed as:

$$\angle G(j\omega) = \angle \left(1 + \frac{j\omega}{z_i}\right) = \tan^{-1} \left(\frac{\omega/z_i}{1}\right)$$
Again using the asymptotic properties, we have for small $\omega$ ($\omega/z_i \ll 1$):

$$\angle G(j\omega) \approx 0$$

and for large $\omega$ ($\omega/z_i \gg 1$) we have:

$$\angle G(j\omega) \approx 90^\circ \text{ or } \frac{\pi}{2}$$

The straight-line approximation of the phase has an initial phase of 0, breaks upward one decade before $z_i$ (or at 0.1$z_i$) at a slope of $45^\circ$/decade, and levels off at one decade after $z_i$ (or at 10$z_i$).

Similar arguments can be made for a pole at $p_i$, except that the magnitude plot for large $\omega$ asymptotes to a straight line with a $-20$dB/decade slope, and the phase angle for large $\omega$ asymptotes to $-90$ or $-\frac{\pi}{2}$. Figure 5 shows the Bode plot for a real pole ($p = 3$) and a real zero ($z = 3$).

**Figure 5: Real Pole/Zero**

![Bode plot for real pole/zero](image)

**Complex Poles/Zeros.** Suppose that we cannot factor an element down to first order terms with real roots. We can at least factor to the corresponding quadratic, and the result is a complex conjugate pair of roots. Suppose we have the following:

$$G(s) = \frac{1}{1 + \left(\frac{2\zeta}{\omega_n}\right)s + \left(\frac{1}{\omega_n^2}\right)s^2}$$

Evaluating at $s = j\omega$ we have:

$$G(j\omega) = \frac{1}{1 + \left(\frac{2\zeta}{\omega_n}\right)j\omega + \left(\frac{1}{\omega_n^2}\right)(j\omega)^2}$$

$$= \frac{1}{\left[1 - \omega^2/\omega_n^2\right] + \left[2\zeta\omega/\omega_n\right]j}$$

We can again look at the asymptotic properties. For small $\omega$, we have:

$$|G(j\omega)|_{dB} \approx -20\log_{10}\sqrt{(1 - 0)^2 + 0^2}$$

$$= -20\log_{10}1 = 0$$
For large $\omega$, we have:

$$|G(j\omega)|_{dB} \approx -20\log_{10}\sqrt{\frac{\omega^4}{\omega_n^4}}$$
$$= -20\log_{10}\frac{\omega^2}{\omega_n^2} = -40\log_{10}\frac{\omega}{\omega_n}$$

The resulting Bode plot for a system with $\omega_n = 5$ and different $\zeta$'s is shown in Figure 6.

Figure 6: Complex Poles

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**Pure Time Delay.** Suppose we have a pure time delay as one of our elements. In this Laplace domain, this shows up as:

$$G(s) = e^{-T_ds}$$

where $T_d$ is the pure delay in seconds. Evaluating at $s = j\omega$ we obtain:

$$G(j\omega) = e^{-T_d j\omega}$$

The corresponding magnitude is:

$$|G(j\omega)|_{dB} = 20\log_{10} 1 = 0$$

and the corresponding phase angle is:

$$\angle G(j\omega) = -T_d \omega$$

So the magnitude plot is a constant 0 for all frequencies, and the phase is a straight line with slope $-T_d$, which passes through $-T_d$ degrees at $\omega = 1$.

**Putting it all together**

So we have learned a bit about the individual components, but how do we put it all together. This is best illustrated with an example. Suppose we have the following system:

$$G(s) = \frac{10(s+10)}{s(s+2)(s+5)}$$
The first step is to put it in the following form:

\[
G(s) = \frac{10(1+s/10) \cdot 10}{s(1+s/2)(1+s/5) \cdot 2 \cdot 5} = \frac{10(1+s/10)}{s(1+s/2)(1+s/5)}
\]

Evaluating at \( s = j\omega \) we have:

\[
G(j\omega) = \frac{10(1+j\omega/10)}{j\omega(1+j\omega/2)(1+j\omega/5)}
\]

Each of the elements of the transfer function are in a form that we recognize. We simply plot each one on the same graph. Let’s first consider the magnitude plot. The gain of 10 has a constant magnitude at 20dB. The zero at 10 has zero magnitude until 10rad/sec, where it breaks upwards at \( +20\text{dB/decade} \). The pole at the origin has a slope of \( -20\text{dB/decade} \), passing through 0dB at 1rad/sec. The pole at 2 has a magnitude of 0dB until 2rad/sec, at which point it breaks downward at \( -20\text{dB/decade} \). The pole at 5 has a magnitude of 0dB until 5rad/sec, at which point it breaks downward at \( -20\text{dB/decade} \). We can simply add up all of these components to get the total magnitude in decibels.

For the phase, we do exactly the same thing. The gain of 10 contributes nothing to the phase. The zero at 10 initially has a phase of \( 0^\circ \), breaks upward at \( 45^\circ/\text{decade} \) at 1rad/sec, and flattens back out at 100rad/sec at \( 90^\circ \). The pole at the origin contributes a constant \( -90^\circ \) over all frequencies. The pole at 2 initially has a phase of \( 0^\circ \), breaks downward at \( -45^\circ/\text{decade} \) at 0.2rad/sec, and flattens back out at 20rad/sec at \( -90^\circ \). The pole at 5 initially has a phase of \( 0^\circ \), breaks downward at \( -45^\circ/\text{decade} \) at 0.5rad/sec, and flattens back out at 50rad/sec at \( -90^\circ \). As with the magnitude plot, we simply add these up to get the total phase profile.

The resulting Bode plots of the components are shown in Figure 7.

If we combine all of the components, the resulting Bode plot is shown in Figure 8.
Figure 8: Composite of the Bode Example