Start by printing your name in the above box and check your section in the box to the left.

Do not detach pages from this exam packet or unstaple the packet.

Please write neatly. Answers which are illegible for the grader cannot be given credit.

Show your work. Except for problems 1-3,8, we need to see details of your computation.

All functions can be differentiated arbitrarily often unless otherwise specified.

No notes, books, calculators, computers, or other electronic aids can be allowed.

You have 90 minutes time to complete your work.

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Problem 1) True/False questions (20 points), no justifications needed

1)  

Every function \( f(x, y) \) of two variables has either a global minimum or a global maximum.

**Solution:**
Take for example \( f(x, y) = x + y \). This function has a constant nonzero gradient and so no critical point. It is unbounded above and below.

2)  

The linearization of the function \( f(x, y) = e^{x+3y} \) at \((0, 0)\) is \( L(x, y) = 1 + x + 3y \).

**Solution:**
Use the definition of linearization. The gradient of \( f \) is \( \nabla f = (e^{x+3y}, 3e^{x+3y}) \). At \((0, 0)\) this is \((1, 3)\). We have \( f(0, 0) = 1 \) so that \( L(x, y) = 1 + x + 3y \).

3)  

The function \( f(x, y, z) = x^2 \cos(z) + x^3 y^2 z + (y - 2)^3 y^5 \) satisfies the partial differential equation \( f_{xyxxy} = 12 \).

**Solution:**
Use Clairiot.

4)  

If \( xe^z = y^2 z \), then \( \partial z / \partial x = e^z / (y^2 - xe^z) \).

**Solution:**
This is a direct application of implicit differentiation \( z_x = -f_x / f_z \).

5)  

The function \( \cos(x^2) \cos(y^2) \) has a local maximum at \((0, 0)\).

**Solution:**
The value at \((0, 0)\) is equal to 1. The functions and so the product take values between \(-1\) and 1.

6)  

The value of the double integral \( \int_0^{\pi/4} \int_0^2 x^3 \cos(y) \, dx \, dy \) is the same as \( (\int_0^{\pi/4} \cos(y) \, dy)(\int_0^2 x^3 \, dx) \).
Solution:
The function $\cos(y)$ is a constant for the inner integral so that we can pull it out of the inner integral.

7) $\text{T}$ $\text{F}$ The gradient of $f(x, y)$ is always tangent to the level curves of $f$.

Solution:
It is perpendicular

8) $\text{T}$ $\text{F}$ If $f(x, y, z) = x - 2y + z$, then the largest possible directional derivative $D_{\vec{v}}f$ at any point in space is $\sqrt{6}$.

Solution:
The gradient has length $\sqrt{6}$. The directional derivative into the direction of the gradient is the length of the gradient.

9) $\text{T}$ $\text{F}$ $\int_0^1 \int_0^1 (x^2 + y^2) \, dx \, dy = \int_0^1 \int_0^1 r^3 \, dr \, d\theta$.

Solution:
While the substitution of the function and the $r$ factor have been done correctly, the region changes. The right integral defines a sector, while the left integral is an integral over the unit square.

10) $\text{T}$ $\text{F}$ It is possible that the directional derivative $D_{\vec{v}}f$ is positive for all unit vectors $\vec{v}$.

Solution:
The directional derivative changes sign if $\vec{v}$ is replaced by $-\vec{v}$.

11) $\text{T}$ $\text{F}$ Using linearization of $f(x, y) = xy$ we can estimate $f(0.999, 1.01) \sim 1 - 0.001 + 0.01 = 1.009$.

Solution:
$L(x, y) = 1 - 1 \cdot 0.001 + 1 \cdot 0.01.$
12) \( \text{T} \) \( \text{F} \)  Given a curve \( \vec{r}(t) \) on a surface \( g(x, y, z) = -1 \), then \( \frac{d}{dt} g(\vec{r}(t)) < 0 \).

**Solution:**
It is zero.

13) \( \text{T} \) \( \text{F} \)  If \( f(x, y) \) has a local minimum at \((0, 0)\) then it is possible that \( f_{xy}(0, 0) > 0 \).

**Solution:**
\( D = f_{xx}f_{yy} - f_{xy}^2 > 0 \) is still possible, if \( f_{xx} \) and \( f_{yy} \) are large. For example \( x^2 + y^2 + xy/10 \) has a local minimum at \((0, 0)\) even so \( f_{xy} > 0 \).

14) \( \text{T} \) \( \text{F} \)  The function \( f(x, y) = -x^8 - 2x^6 - y^8 \) has a local minimum at \((0, 0)\).

**Solution:**
One can not use the second derivative test because the discriminant is zero. But the function is zero at \((0, 0)\) and strictly negative everywhere else. Therefore, \((0, 0)\) is a global maximum. It is definitly not a minimum.

15) \( \text{T} \) \( \text{F} \)  If \( \vec{r}(t) \) is a curve in space and \( f \) is a function of three variables, then \( \frac{d}{dt} f(\vec{r}(t)) = 0 \) for \( t = 0 \) implies that \( \vec{r}(0) \) is a critical point of \( f(x, y, z) \).

**Solution:**
We can have \( r(t) = (t, 0, 0) \) and \( f(x, y, z) = x^2 + (y - 1)^2 \).

16) \( \text{T} \) \( \text{F} \)  Let \( a, b, c \) be the number of saddle points, maxima and minima of a function \( f(x, y) \). Then \( a \leq b + c \).

**Solution:**
Already \( x^2 - y^2 \) is a counter example.

17) \( \text{T} \) \( \text{F} \)  If \( f(x, y) \) is a nonzero function of two variables and \( R \) is a region, then \( \int_R f(x, y) \, dx \, dy \) is the volume under the graph of \( f \) and therefore a positive value.

**Solution:**
if \( f \) is replaced by \(-f\), then the sign of the integral changes too.
18) T F We extremize $f(x, y)$ under the constraint $g(x, y) = c$ and obtain a solution $(x_0, y_0)$. If the Lagrange multiplier $\lambda$ is positive, then the solution is a minimum.

Solution:
There is no relation between the sign of $\lambda$ and minima and maxima. Change $g = c$ to $-g = -c$ and the sign of $\lambda$ changes.

19) T F The tangent plane to a surface $f(x, y, z) = 1$ intersects the surface in exactly one point.

Solution:
Take a one sheeted hyperboloid.

20) T F Let $\vec{v}$ be a vector of length 1 in space. Given a function $f(x, y, z)$ of three variables. If $(x_0, y_0, z_0)$ is a critical point of $f$, then it is a critical point of $g(x, y, z) = D_v f(x, y, z)$.

Solution:
Let $\vec{v} = (1, 0, 0)$. Now $g(x, y, z) = f_x(x, y, z)$ and $\nabla g = \langle f_{xx}, f_{xy}, f_{xz} \rangle$.

Problem 2) (10 points)

a) (6 points) Match the regions with the corresponding double integrals
Enter a, b, c, d, e or f

Integral of \( f(x, y) \)

\[
\int_0^1 \int_{\sqrt{x}}^x f(x, y) \, dy \, dx
\]

\[
\int_0^1 \int_{\sqrt{y}}^0 f(x, y) \, dx \, dy
\]

\[
\int_0^1 \int_{y}^1 f(x, y) \, dx \, dy
\]

Enter a, b, c, d, e or f

Integral of \( f(x, y) \)

\[
\int_0^1 \int_{\sqrt{1-x^2}}^0 f(x, y) \, dy \, dx
\]

\[
\int_0^1 \int_{(1-x)^2}^1 f(x, y) \, dy \, dx
\]

\[
\int_0^1 \int_{(1-x)^2}^{\sqrt{1-x^2}} f(x, y) \, dy \, dx
\]

b) (4 points) Match the PDE’s with the names. No justifications are needed.

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<tr>
<th>Enter A, B, C, D here</th>
<th>PDE</th>
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<tr>
<td>( f_{xx} = -f_{yy} )</td>
<td>( f_{xx} = -f_{yy} )</td>
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A) Wave equation  B) Heat equation  C) Transport equation  D) Laplace equation

Solution:

a) b
a) c f
d e
D A
b) C B

Problem 3) (10 points)
a) (3 points) Find and classify all the critical points of \( f(x, y) = xy - x \) on the plane.

b) (2 points) Decide whether an absolute maximum or an absolute minimum of \( f \) exists on the plane \( \mathbb{R}^2 \).

c) (3 points) Use the method of Lagrange multipliers to find the maximum and minimum of \( f \) on the boundary \( x^2 + 4y^2 = 12 \) of the elliptical region \( G : x^2 + 4y^2 \leq 12 \).

d) (2 points) Find the absolute maximum and absolute minimum of \( f \) on the region \( G \) given in c).

Solution:

a) \( \nabla f = \langle y - 1, x \rangle = \vec{0} \) for \( (x, y) = (0, 1) \). Since \( f_{xx} = f_{yy} = 0 \) and \( f_{xy} = 1 \), the discriminant is \( D = 0^2 - 2^2 < 0 \) and \( (0, 1) \) is a saddle point.

b) There is no global maximum, nor any global minimum on the plane. On the \( x \)-axes \( y = 0 \) for example, we have \( f(x, 0) = -x \) which is unbounded both from above and from below.

c) The Lagrange equations are

\[
\begin{align*}
    y - 1 &= \lambda \cdot 2x \\
    x &= \lambda \cdot 8y \\
    x^2 + 4y^2 &= 12.
\end{align*}
\]

\( y \neq 0 \), because otherwise the second equation would give \( x = 0 \), contradicting the constraint. Also \( x \neq 0 \), because otherwise, the first equation would give \( y = 1 \), again contradicting the constraint. Dividing the first by the second gives \( (y - 1)/x = (1/4)x/y \) or \( 4y(y - 1) = x^2 \). Plugging this into the constraint gives \( 4y(y - 1) + 4y^2 = 12 \). The solutions of this quadratic equation are \( y = 3/2 \) or \( y = -1 \). The extrema are \( (\pm 2\sqrt{2}, -1) \) and \( (\pm \sqrt{3}, 1.5) \).

Since \( f(2\sqrt{2}, -1) = -4\sqrt{2} \), \( f(-2\sqrt{2}, -1) = 4\sqrt{2} \), \( f(\sqrt{3}, 1.5) = \frac{\sqrt{3}}{2} \) and \( f(-\sqrt{3}, 1.5) = -\frac{\sqrt{3}}{2} \), the maximum is \( (x, y) = (-2\sqrt{2}, -1) \) and the minimum is \( (x, y) = (2\sqrt{2}, -1) \).

d) From parts (a) and (c) we have a list of all candidates for global extrema. The global maximum value of \( f \) on \( G \) is \( f(-2\sqrt{2}, -1) = 4\sqrt{2} \), the global minimal value on \( G \) is \( f(2\sqrt{2}, -1) = -4\sqrt{2} \).

Problem 4) (10 points)

Find the cylindrical basket which is open on the top has has the largest volume for fixed area \( \pi \). If \( x \) is the radius and \( y \) is the height, we have to extremize \( f(x, y) = \pi x^2 y \) under the constraint \( g(x, y) = 2\pi xy + \pi x^2 = \pi \). Use the method of Lagrange multipliers.
Solution:
The Lagrange equations are
\[ 2xy\pi = (2x\pi + 2y\pi)\lambda \]
\[ \pi x^2 = 2\pi x\lambda \]
\[ \pi x^2 + 2\pi xy = \pi \]

Since \( x = 0 \) is not possible (it would violate the constraint), we can divide the second equations by \( x \) and divide the first by the second equation. This gives \( x = y = 1/\sqrt{3} \).

The maximum value is \( \pi\sqrt{3}/9 \).

Problem 5) (10 points)

The Pac-Man region \( R \) is bounded by the lines \( y = x, y = -x \) and the unit circle. The number
\[ a = \frac{\int \int_R x \, dx \, dy}{\int \int_R 1 \, dx \, dy} \]
defines the point \( C = (a, 0) \) called center of mass of the region. Find it.
Solution:

\[
\begin{align*}
\int_{\pi/4}^{7\pi/4} \int_0^1 r \cos(\theta) \, r \, dr \, d\theta &= (1/3) \sin(\theta) \bigg|_{\pi/4}^{7\pi/4} = -\sqrt{2}/3. \\
\int_{\pi/4}^{7\pi/4} \int_0^1 r \, dr \, d\theta &= (1/2)(7\pi/4 - \pi/4) = 6\pi/8 = 3\pi/4.
\end{align*}
\]

The second integral is the area of the Pac-Man, which is \(3/4\) of the area of the full disc. Dividing the first by the second integral gives the result \(a = -4\sqrt{2}/(9\pi)\). The center of mass is \((-4\sqrt{2}/(9\pi), 0)\).

Problem 6) (10 points)

a) (5 points) Find the tangent plane to the surface \(\sqrt{xyz} = 60\) at \((x, y, z) = (100, 36, 1)\).

b) (5 points) Estimate \(\sqrt{100.1 \times 36.1 \times 0.999}\) using linear approximation. Here, for clarity reasons, we use \(*\) for the usual multiplication for numbers.

Solution:

a) We have

\[
\nabla f(x, y, z) = \left(\sqrt{yz}/x, \sqrt{xz}/y, \sqrt{xy}/z\right) / 2
\]

\[
\nabla f(100, 36, 1) = \left(6/10, 10/6, 60\right) / 2
\]

The tangent plane is \((3/10)x + (5/6)y + 30z = 90\). We have obtained the constant on the right by plugging in the point \((x, y, z) = (100, 36, 1)\).

b) Since \(f(100, 36, 1) = 60\), we have \(L(x, y, z) = 60 + (3/10)(x-100) + (5/6)(y-36) + 30(z-1)\). We have \(L(100.1, 36.1, 0.999) = 60 + 0.03 + 0.08333.. - 0.03 = 60.08333.. = 60 + 1/12\). This is very close to the actual value 60.0832455.... You have in this problem computed the square root of a real number by hand with an accuracy of 4 digits after the comma.

Problem 7) (10 points)

Oliver got a diagmagnetic kit, where strong magnets produce a force field in which pyrolytic graphic flots. The gravitational field produces a well of the form \(f(x, y) = x^4 + y^3 - 2x^2 - 3y\). Find all critical points of this function and classify them. Is there a global minimum?
Solution:
To find the critical points, we have to solve the system of equations $f_x = 4x^3 - 4x = 0, f_y = 3y^2 - 3 = 0$. The first equation gives $x = 0$ or $x = \pm 1$. The second equation $f_y = 3y^2 - 3 = 0$ gives $y = \pm 1$. There are $3 \cdot 2 = 6$ critical points. We compute the discriminant $D = 6y(12x^2 - 4)$ and $f_{xx} = 12x^2 - 4$ at each of the 6 points and use the second derivative test to determine the nature of the critical point.

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<th>$D$</th>
<th>$f_{xx}$</th>
<th>nature</th>
<th>value</th>
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<td>-48</td>
<td>8</td>
<td>saddle</td>
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<td>(1, 1)</td>
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<td>8</td>
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There is no global minimum, nor any global maximum since for $x = 0$, the function is $f(0, y) = y^3 - 3y$ which is unbounded from above and from below (it goes to $\pm \infty$ for $y \to \pm \infty$).

Problem 8) (10 points)

Let $f(x, y) = xy$.

a) (2 points) Find the direction of maximal increase at the point $(1, 1)$.

b) (3 points) Find the directional derivative at $(1, 1)$ in the direction $\langle 3/5, 4/5 \rangle$.

c) (2 points) The curve $\vec{r}(t) = \langle \sqrt{2}\sin(t), \sqrt{2}\cos(t) \rangle$ passes through the point $(1, 1)$ at some time $t_0$. Find $\frac{d}{dt}f(\vec{r}(t))$ at time $t_0$ directly.

d) (3 points) Find $\frac{d}{dt}f(\vec{r}(t))$ at time $t_0$ using the multivariable chain rule.
Solution:

a) $\nabla f(x, y) = \langle y, x \rangle$, $\nabla f(1, 1) = \langle 1, 1 \rangle$. The direction of maximal increase is $\langle 1, 1 \rangle / \sqrt{2}$.

b) $D_v f(1, 1) = \langle 1, 1 \rangle \cdot \langle 3/5, 4/5 \rangle = 7/5$.

c) It is at the time $t_0 = \pi/4$, where the curve passes through the point $(1, 1)$. We have $f(\vec{r}(t)) = 2 \cos(t) \sin(t) = \sin(2t)$ and $d/dt f(\vec{r}(t)) = 2 \cos(2t)$ which is $0$ at time $t = \pi/4$.

d) By the multi variable chain rule, $\frac{d}{dt} f(\vec{r}(t)) = \nabla f(1, 1) \cdot \langle -\sin(\pi/4), \cos(\pi/4) \rangle = 0$.

Problem 9) (10 points)

Integrate the function

$$f(x, y) = \frac{y^5 - 1}{y^{1/3} - y^{1/4}}$$

on the finite region bounded by the curves $y = x^3$ and $y = x^4$.

Solution:

Make a picture! The two graphs intersect at 0 and 1 forming a grass shaped region.

The type I integral

$$\int_0^1 \int_{x^3}^{x^4} \frac{y^5 - 1}{y^{1/3} - y^{1/4}} \, dy \, dx$$

can not be evaluated (at least not without going through difficult substitution/partial fraction procedures which can fill pages).

We decide therefore, to change the order of integration and write a type II integral:

$$\int_0^1 \int_{y^{1/3}}^{y^{1/4}} \frac{y^5 - 1}{y^{1/3} - y^{1/4}} \, dx \, dy$$

Now the inner integral can be solved and give $(1 - y^5)$. We end up with $\int_0^1 (1 - y^5) \, dy = 5/6$. 

Problem 10) (10 points)

The main building of a mill has a cone shaped roof and cylindrical walls. If the cylinder has radius $r$, the height of the side wall is $h$ and the height of the roof is $h$, then the volume is

$$V(h, r) = \pi r^2 h + h\pi r^2/3 = (4\pi/3)hr^2$$

and assume the cost of the building is

$$A(h, r) = \pi r^2 + 2\pi rh + \pi 2r^2 = \pi(3r^2 + 2rh)$$

which is the area of the ground plus the area of the wall plus $2\pi rh$, the cost for the roof. For fixed volume $V(h, r) = 4\pi/3$, minimize the cost $A(h, r)$ using the Lagrange multiplier method.

Solution:
After dividing out some constants and taking $g = hr^2 = 1$, the Lagrange equations become

$$
\begin{align*}
6r + 2h &= \lambda 2hr \\
2r &= \lambda r^2 \\
r^2h &= 1
\end{align*}
$$

The second equation can be divided by $r$ since $r = 0$ is incompatible with the third equation. The first can be divided by 2. We get

$$
\begin{align*}
3* r + h &= \lambda hr \\
2 &= \lambda r \\
r^2h &= 1
\end{align*}
$$

You can plug in $\lambda r$ from the second equation into the first to get

$$
\begin{align*}
3r + h &= 2h \\
r^2h &= 1
\end{align*}
$$

The first equation shows $h = 3r$ and plugging this into the third equation gives $r = 1/3^{1/3}$ and $h = 3r = 3^{2/3}$.