Distances overview

DISTANCE POINT-POINT (3D). If $P$ and $Q$ are two points, then

$$d(P, Q) = |PQ|$$

is the distance between $P$ and $Q$. We use the notation $|\vec{v}|$ instead of $||\vec{v}||$ in this handout.

DISTANCE POINT-PLANE (3D). If $P$ is a point in space and $\Sigma : \vec{n} \cdot \vec{x} = d$ is a plane containing a point $Q$, then

$$d(P, \Sigma) = \frac{|PQ \cdot \vec{n}|}{|\vec{n}|}$$

is the distance between $P$ and the plane. Proof: you can see this as a scalar projection of $PQ$ onto $\vec{n}$. (P.S. If the plane is parametrized $\vec{r} = \vec{O}Q + t\vec{v} + s\vec{w}$, find first $\vec{n} = \vec{v} \times \vec{w}$.)

DISTANCE POINT-LINE (3D). If $P$ is a point in space and $L$ is the line $\vec{r}(t) = \vec{0}Q + t\vec{u}$, then

$$d(P, L) = \frac{|(PQ) \times \vec{u}|}{|\vec{u}|}$$

is the distance between $P$ and the line $L$. Proof: the area divided by base length is height of parallelogram.

DISTANCE LINE-LINE (3D). $L$ is the line $\vec{r}(t) = Q + t\vec{u}$ and $M$ is the line $\vec{s}(t) = P + t\vec{v}$, then

$$d(L, M) = \frac{|(PQ) \cdot (\vec{u} \times \vec{v})|}{|\vec{u} \times \vec{v}|}$$

is the distance between the two lines $L$ and $M$. Proof: the distance is the length of the vector projection of $PQ$ onto $\vec{u} \times \vec{v}$ which is normal to both lines.

DISTANCE PLANE-PLANE (3D). If $\vec{n} \cdot \vec{x} = d$ and $\vec{n} \cdot \vec{x} = e$ are two parallel planes, then their distance is

$$\frac{|e - d|}{|\vec{n}|}.$$

Non-parallel planes have distance 0. Proof: use the distance formula between point and plane.
**EXAMPLES**

DISTANCE POINT-POINT (3D). \( P = (-5, 2, 4) \) and \( Q = (-2, 2, 0) \) are two points, then

\[
d(P, Q) = |\vec{PQ}| = \sqrt{(-5 + 2)^2 + (2 - 2)^2 + (0 - 4)^2} = 5.
\]

A question: what is the distance between the point \((-5, 2, 4)\) and the sphere \((x + 2)^2 + (y - 2)^2 + z^2 = 1)\?

DISTANCE POINT-PLANE (3D). \( P = (7, 1, 4) \) is a point and \( \Sigma : 2x + 4y + 5z = 9 \) is a plane which contains the point \( Q = (0, 1, 1) \). Then

\[
d(P, \Sigma) = \frac{|\langle -7, 0, -3 \rangle \cdot \langle 2, 4, 5 \rangle|}{|\langle 2, 4, 5 \rangle|} = \frac{29}{\sqrt{45}}
\]

is the distance between \( P \) and \( \Sigma \). A Question: without the absolute value, the result would have been negative. What does this tell about the point \( P \)?

DISTANCE POINT-LINE (3D). \( P = (2, 3, 1) \) is a point in space and \( L \) is the line \( \vec{r}(t) = (1, 1, 2) + t(5, 0, 1) \). Then

\[
d(P, L) = \frac{|\langle -1, -2, 1 \rangle \times \langle 5, 0, 1 \rangle|}{\langle 5, 0, 1 \rangle} = \frac{|\langle -2, 6, 10 \rangle|}{\sqrt{26}} = \frac{\sqrt{140}}{\sqrt{26}}
\]

is the distance between \( P \) and \( L \). Question to the reader: what is the equation of the plane which contains the point \( P \) and the line \( L \)?

DISTANCE LINE-LINE (3D). \( L \) is the line \( \vec{r}(t) = (2, 1, 4) + t(-1, 1, 0) \) and \( M \) is the line \( \vec{s}(t) = (-1, 0, 2) + t(5, 1, 2) \). The cross product of \( \langle -1, 1, 0 \rangle \) and \( \langle 5, 1, 2 \rangle \) is \( \langle 2, 2, -6 \rangle \). The distance between these two lines is

\[
d(L, M) = \frac{|\langle 3, 1, 2 \rangle \cdot \langle 2, 2, -6 \rangle|}{|\langle 2, 2, -6 \rangle|} = \frac{4}{\sqrt{44}}.
\]

Question to the reader: also here, without the absolute value, the formula can give a negative result. What happens with this sign, when \( P \) and \( Q \) are interchanged?

DISTANCE PLANE-PLANE (3D). \( 5x + 4y + 3z = 8 \) and \( 5x + 4y + 3z = 1 \) are two parallel planes. Their distance is

\[
\frac{|8 - 1|}{|\langle 5, 4, 3 \rangle|} = \frac{7}{\sqrt{50}}.
\]

Question for the reader: what is the distance between the planes \( x + 3y - 2z = 2 \) and \( 5x + 15y - 10z = 30 \)?