1. Find the kernel of the transformation $x \rightarrow Ax$, then write down a set of vectors which span the image of $A$.
   
   a) $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 & 5 & 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 & 5 & 4 & 3 & 2 & 1 \end{bmatrix}$, b) $\begin{bmatrix} 4 & 2 & 1 & 2 & 4 \\ 4 & 2 & 1 & 2 & 4 \\ 4 & 2 & 1 & 2 & 4 \\ 4 & 2 & 1 & 2 & 4 \end{bmatrix}$, 
   
   c) $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$, d) $\begin{bmatrix} 1 \\ 3 \\ 4 \\ 5 \end{bmatrix}$.

2. a) Give an example of a transformation from $\mathbb{R}^6$ to $\mathbb{R}^3$ for which the image is spanned by the two vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$.
   
   b) Express the kernel of the $1 \times 4$ matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$ as the image of a $4 \times 3$ matrix $B$.

3. a) What is the image and kernel of the shear $\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$?
   
   b) What is the image and kernel of the rotation–dilation $\begin{bmatrix} 5 & 12 \\ -12 & 5 \end{bmatrix}$?
   
   c) What is the image and kernel of the projection $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$?
   
   d) What is the image and kernel of the reflection $\frac{1}{13} \begin{bmatrix} 5 & 12 \\ 12 & -5 \end{bmatrix}$?
e) What is the image and kernel of the matrix
\[
A = \begin{bmatrix}
1 & 2 \\
3 & 4 \\
5 & 6
\end{bmatrix}.
\]

4 These problems are a bit more abstract. You might want to make use of problem sessions, office hours and MQC. Let \( A \) be an arbitrary \( 5 \times 5 \) matrix and \( B = \text{rref}(A) \).

a) Is it true that \( \text{im}(A) = \text{im}(B) \)? Explain why or why not.

b) Is it true that \( \ker(A) = \ker(B) \)? Explain why or why not.

c) Be creative and find a \( 3 \times 3 \) or \( 4 \times 4 \) matrix for which \( \text{im}(A) = \ker(A) \).

5 Let \( A \) be a \( n \times n \) matrix. Is \( X \subset Y \) or is \( Y \subset X \)?

a) \( X = \text{im}(A) \) and \( Y = \text{im}(A^3) \)?

b) \( X = \ker(A) \) and \( Y = \ker(A^3) \)?

c) \( X = \ker(A) \) and \( Y = \ker(A^3 + A^2) \)?

d) \( X = \text{im}(A) \) and \( Y = \text{im}(A^3 + A^2) \)?

**Image and kernel**

The **kernel** is the set of vectors \( x \) which satisfy \( Ax = 0 \).

The **image** of a linear map \( x \to Ax \) is the set of all vectors \( Ax \).

The columns of \( A \) **span** the image of \( A \). Every \( x \in \text{im}(A) \) can be written as a linear combination of column vectors.

The image and kernel are both linear spaces: they are closed under addition, scalar multiplication and contain the zero vector. The kernel of a \( n \times n \) matrix is \( \{0\} \) if and only if \( A \) is invertible if and only if the image is \( \mathbb{R}^n \) if and only if \( \text{rref}(A) = 1 \).