1. Which of the following sets are linear spaces? Check in each case the three properties characterizing a linear space. Only a brief explanation is needed (can be a picture too):
   a) \( W = \{(x, y, z) \mid x > 0\} \)
   b) \( W = \{(x, y, z) \mid xyz = 0\} \)
   c) \( W = \{(x, y, z) \mid x = 2y = 3z\} \)
   d) \( W = \{(x, y, z) \mid x = y = z + 1\} \)
   e) \( W = \{(x, y, z) \mid x^2 + y^2 - z^2 = 0\} \)
   f) \( W = \{(x, y, z) \mid x, y, z \text{ are rational numbers}\} \)
   g) \( W = \{(x, y, z) \mid x = y = z = 0\} \)

2. a) Write the three dimensional space \( x + 2y + 3z + 4t = 0 \) as a kernel of a \( 1 \times 4 \) matrix.
   b) Write the same plane as the image of a \( 4 \times 3 \) matrix.
   c) Find a basis for this space.

3. Check whether the given set of vectors is linearly independent
   a) \( \left\{ \begin{bmatrix} 3 \\ 3 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ 6 \end{bmatrix} \right\} \)
   b) \( \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \)
   c) \( \left\{ \begin{bmatrix} 3 \\ 16 \end{bmatrix}, \begin{bmatrix} 4 \\ 18 \end{bmatrix}, \begin{bmatrix} 5 \\ 19 \end{bmatrix} \right\} \)

4. Find a basis for the image as well as as a basis for the kernel of the following matrices
   a) \( \begin{bmatrix} 7 & 0 & 7 \\ 2 & 3 & 8 \\ 9 & 0 & 9 \\ 5 & 6 & 17 \end{bmatrix} \)
   b) \( \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \)
   c) \( \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \)

5. The orthogonal complement of a subspace \( V \) of \( R^n \) is the set \( V^\perp \) of all vectors in \( R^n \) that are perpendicular to every single vector in \( V \). Find a basis for the orthogonal complement in each case:
   a) The line \( L \) in \( R^5 \) spanned by \( \begin{bmatrix} 1 & 2 & 2 & 1 & 1 \end{bmatrix}^T \). (If \( v \) is a row
vector $v^T$ denotes the corresponding column vector).

b) The plane $\Sigma$ in $\mathbb{R}^4$ spanned by $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$ and $\begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix}^T$.

c) The space $V = \{(0, 0)\}$ in the two-dimensional plane $\mathbb{R}^2$.

**Basis**

$V$ is a **linear space** if 0 is in $V$, if $v + w$ is in $V$ for all $v, w$ in $V$ and if $\lambda v$ is in $V$ for every $v$ in $V$ and every $\lambda$ in $\mathbb{R}$. Examples: kernels $V = \ker(A)$ or images $V = \text{im}(A)$ are linear spaces. If $V, W$ are linear spaces and $V$ is a subset of $W$, then $V$ is called a **linear subspace** of $W$. A line through the origin for example is a linear subspace of $\mathbb{R}^3$. A set $\mathcal{B}$ of vectors $\{v_1, \ldots, v_n\}$ **spans** $V$ if every $v \in V$ is a sum of vectors in $\mathcal{B}$. A set $\mathcal{B}$ is linear independent if $a_1v_1 + \cdots + a_nv_n = 0$ implies $a_1 = \cdots = a_n = 0$. It is a **basis** of $V$ if it both spans $V$ and is linearly independent. Example: the standard basis vectors $\{e_1, \ldots, e_n\}$ form a basis of $\mathbb{R}^n$.

How do we determine whether a set of vectors is a basis of $\mathbb{R}^n$? Place the vectors of $\mathcal{B}$ as columns in a matrix $A$, then row reduce $A$. If every column of a matrix has a leading 1, then the set of column vectors $\mathcal{B}$ are linearly independent and the kernel of $A$ is $\{0\}$. How do we determine whether a set of vectors is linearly independent? Place the vectors as columns of a matrix and row reduce. If there is no free variable, then we have linear independence. Example: three vectors in $\mathbb{R}^3$ are linearly independent if they are not in a common plane.