Homework 22: Stability

This homework is due on Friday, March 30, respectively on Tuesday, April 3, 2018.

1 Determine the stability of the dynamical system $x(t+1) = Ax(t)$:
   a) $A = \begin{bmatrix} 0.2 & 0.3 & 0.4 \\ 0.2 & 0.3 & 0.4 \\ 0.9 & 2 & 3 \end{bmatrix}$.
   b) $A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & -2 & 1 \end{bmatrix}$.

2 For which constants $a$ is the system $x(t+1) = Ax(t)$ stable?
   a) $A = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}$.
   b) $A = \begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix}$.
   c) $A = \begin{bmatrix} a & a \\ a & a \end{bmatrix}$.

3 For which real values $k$ does the drawing rule
   
   $x(t+1) = x(t) - ky(t)$
   $y(t+1) = y(t) + kx(t+1)$

   produce trajectories which are ellipses? Write the system first as a discrete dynamical system $v(t+1) = Av(t)$ and look for the $k$ for which the eigenvalues $\lambda_k$ satisfy $|\lambda_k| = 1$.

4 Find the eigenvalues of
   
   $A = \begin{bmatrix} 0 & a & b & c & 0 & 0 \\ 0 & 0 & a & b & c & 0 \\ 0 & 0 & 0 & a & b & c \\ c & 0 & 0 & 0 & a & b \\ b & c & 0 & 0 & 0 & a \\ a & b & c & 0 & 0 & 0 \end{bmatrix}$
Where \(a, b\) and \(c\) are arbitrary constants. Verify that the discrete dynamical system is stable for \(|a| + |b| + |c| < 1\).

5 In the following, answer each question with a short explanation. We say \(A\) is stable if the origin \(\vec{0}\) is a stable equilibrium.

a) True or false: the identity matrix is stable.
b) True or false: the zero matrix is stable.
c) True or false: every horizontal shear is stable.
d) True or false: any reflection matrix is stable.
e) True or false: \(A\) is stable if and only if \(A^T\) is stable.
f) True or false: \(A\) is stable if and only if \(A^{-1}\) is stable.
g) True or false: \(A\) is stable if and only if \(A + 1\) is stable.
h) True or false: \(A\) is stable if and only if \(A^2\) is stable.
i) True or false: \(A\) is stable if \(A^2 = 0\).
j) True or false: \(A\) is unstable if \(A^2 = A\).
k) True or false: \(A\) is stable if \(A\) is diagonalizable.

### Stability

A discrete dynamical system \(x(t + 1) = Ax(t)\) is **asymptotically stable** if \(x(t) \to 0\) as \(t \to \infty\) for all initial conditions \(x(0)\). (If we say ”stable” we always mean asymptotically stable). The main result covered in this section is that a system is asymptotically stable if and only all eigenvalues of \(A\) have absolute value \(|\lambda_j| < 1\). For example, a rotation dilation \(A\) with first column \(Ae_1 = \begin{bmatrix} a \\ b \end{bmatrix}\) is stable if and only if \(a^2 + b^2 < 1\). We often just say “\(A\) is stable” rather than “the origin is stable for the discrete dynamical system \(x \mapsto Ax\)”.