Homework 23: Symmetric matrices

This homework is due on Monday, April 2, respectively on Tuesday, April 3, 2018.

1. Give a reason why it’s true or provide a counterexample.
   a) The product of two symmetric matrices is symmetric.
   b) The sum of two symmetric matrices is symmetric.
   c) The sum of two anti-symmetric matrices is anti-symmetric.
   d) The inverse of an invertible symmetric matrix is symmetric.
   e) If $B$ is an arbitrary $n \times m$ matrix, then $A = B^T B$ is symmetric.
   f) If $A$ is similar to $B$ and $A$ is symmetric, then $B$ is symmetric.
   g) $A = SBS^{-1}$ with $S^T S = I_n$, $A$ symmetric $\Rightarrow B$ is symmetric.
   h) Every symmetric matrix is diagonalizable.
   i) Only the zero matrix is both anti-symmetric and symmetric.

2. Find all the eigenvalues and eigenvectors of the matrix

   $$A = \begin{bmatrix}
   2019 & 2 & 3 & 4 & 5 \\
   2 & 2022 & 6 & 8 & 10 \\
   3 & 6 & 2027 & 12 & 15 \\
   4 & 8 & 12 & 2034 & 20 \\
   5 & 10 & 15 & 20 & 2043 \\
   \end{bmatrix}.$$ 

3. a) Find the eigenvalues and orthonormal eigenbasis of $A = \begin{bmatrix}1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$.

   b) Find $\det(\begin{bmatrix}7 & 2 & 2 & 2 & 2 \\ 2 & 7 & 2 & 2 & 2 \\
   2 \end{bmatrix})$ using eigenvalues.
Group the matrices which are similar.

\[
A = \begin{bmatrix}
  2 & 1 & 0 & 0 \\
  0 & 2 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
\end{bmatrix} \quad B = \begin{bmatrix}
  1 & 1 & 0 & 0 \\
  1 & 1 & 0 & 0 \\
  0 & 0 & 1 & 1 \\
  0 & 0 & 1 & 1 \\
\end{bmatrix} \quad C = \begin{bmatrix}
  2 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 2 \\
  0 & 0 & 1 & 0 \\
\end{bmatrix} \quad D = \begin{bmatrix}
  1 & 0 & 1 & 0 \\
  0 & 1 & 0 & 1 \\
  1 & 0 & 1 & 0 \\
  0 & 1 & 0 & 1 \\
\end{bmatrix}
\]

Find the eigenvalues and eigenvectors of the Laplacian of the Bunny graph. The Laplacian of a graph with \( n \) nodes is the \( n \times n \) matrix \( A \) which for \( i \neq j \) has \( A_{ij} = -1 \) if \( i, j \) are connected and 0 if not. The diagonal entries \( A_{ii} \) are chosen so that each row add up to 0.

\[
A = \begin{bmatrix}
  2 & -1 & -1 & 0 & 0 \\
  -1 & 2 & -1 & 0 & 0 \\
  -1 & -1 & 4 & -1 & -1 \\
  0 & 0 & -1 & 1 & 0 \\
  0 & 0 & -1 & 0 & 1 \\
\end{bmatrix}
\]

Symmetric matrices

\( A \) is **symmetric** if \( A^T = A \) and **anti-symmetric** if \( A^T = -A \). Projections or reflections are symmetric. Symmetric matrices appear in physics or statistics: observables like energy, position or momentum matrices are symmetric, correlation matrices are symmetric. In multi-variable calculus the Hessian matrix consisting of the second derivatives is symmetric. The spectral theorem tells that a symmetric matrix has real eigenvalues, that it has an orthonormal eigenbasis and that can be diagonalized as \( B = S^{-1}AS \) with an orthogonal matrix \( S \).