Homework 27: Differential operators

This homework is due on Friday, April 13, respectively on Tuesday, April 17, 2018.

The linear spaces $C^\infty$, $C_{\text{per}}^\infty$, $P$ and $T$ are defined on the next page.

1. The linear map $Df(x) = f'(x)$ is an example of a **differential operator**. It has the constant functions as the kernel. This means that there is no unique inverse. One inverse is $S f(x) = D^{-1} f(x) = \int_0^x f(t) \, dt$.
   a) Evaluate $D \sin$, $D \cos$, $D \tan$, $S 1/(1 + x^2)$, $S \tan$.
   b) Can you find an eigenfunction (= eigenvector) $f$ of $D$ to the eigenvalue $-101$?
   c) Verify that if $f$ is an eigenfunction of $D$ to the eigenvalue $2$, then $f$ is also an eigenfunction of $D^4 - 2D + 77$. What is the eigenvalue?

2. a) Find a solution of the equation $D^2 f = 2x + 1/x$ on the space $C^\infty((0, \infty))$ of all smooth functions on the positive real axes.
   b) Find two linearly independent solutions of the eigenvalue equation $D^2 f = -10'000 f$ on the space $C_{\text{per}}^\infty$.

3. a) Find a basis for the kernel of $D^3$ on the linear space $P$ of polynomials.
   b) Find the image $D^3 + D + 1$ on the linear space $P$?
   c) Find the eigenvalues of $D^3 + D + 1$ on the space $C_{\text{per}}^\infty$ of smooth periodic functions with period $2\pi$.
   d) Find the kernel of $Af = (D - \sin(t)) f(t)$ on $C_{\text{per}}^\infty$.

4. a) Check that $Q f(x) = xf(x)$ and $P f(x) = iD f(x)$ satisfy the Heisenberg commutation relation: $(PQ - QP) f = i f$.
   b) Check that for any real $\omega$, the function $e^{i\omega t}$ is an eigenfunction of $iD$ in $C^\infty$.
   c) Check that on $C_{\text{per}}^\infty$, only the functions $e^{i\omega t}$ with integer $\omega$ are eigenfunctions. (Momentum $\omega$ is quantized.)
a) Verify that \( Sf(x) = \int_0^x f(t) \, dt \) is a linear operator on the linear space \( C^\infty \) of smooth functions.
b) Show that \( DSf(x) = f(x) \) and c) show that \( SDf(x) = f(x) - f(0) \). What is the theorem?

## Differential operators

A function is **smooth** if it can be differentiated arbitrarily often. The space \( C^\infty \) of real valued smooth functions is a linear space: if \( f, g \) are in \( C^\infty \), then \( f + g \), the zero function 0 is in \( C^\infty \) and \( \lambda f \) is in \( C^\infty \) for every real \( \lambda \). \( C^\infty \) contains the linear space \( P \) of all **polynomials**. The space \( C^\infty_{\text{per}} \) of smooth periodic functions with period \( 2\pi \) forms a linear space too. It contains the linear subspace \( T \) of trigonometric polynomials. The space \( P \) of polynomials is spanned by \( \{1, x, x^2, x^3, \ldots\} \) and the space \( T \) of trigonometric polynomials is spanned by \( \{1, \cos(x), \sin(x), \cos(2x), \sin(2x), \ldots\} \). They are infinite dimensional. The space \( P_3 \) of cubic polynomials \( d + cx + bx^2 + ax^3 \) is 4-dimensional as it has the basis \( \{1, x, x^2, x^3\} \). The transformation map \( D : f \rightarrow f' \) is linear: it satisfies \( D(f + g) = Df + Dg \), \( D(\lambda f) = \lambda Df \) and \( D0 = 0 \). We call any polynomial of \( D \) like \( D^2 - D + 1 \) a **differential operator**. The linear map \( D \) on \( C^\infty \) has as the kernel the one dimensional space of constant functions. What are the eigenvalues and eigenvectors of \( D \)? Because \( De^{\lambda x} = \lambda e^{\lambda x} \), every real number \( \lambda \) is an eigenvalue on \( C^\infty \). The linear map \( D \) has no real eigenvalues on \( C^\infty_{\text{per}} \) but complex eigenvalues \( in \) as \( De^{inx} = ine^{inx} \), where \( n \) is an integer. The fact that they are quantized is the reason why quantum mechanics is called “quantum” (the operator \( P = iD \) is called “momentum”) and \( Qf = xf \) “position”). The square \(-D^2\) has now real eigenvalues \( n^2 \), where \( n \) is an integer. It is the energy operator of a particle on the circle. The eigenfunctions are \( 1, \sin(nx) \) and \( \cos(nx) \). We are interested in \( D \) because it will allow us to solve differential equations like \( (D^2 + 5D + 6)f = \sin(5x) \).