# Homework 31: Partial differential equations

This is the last homework. It is due on the last day of classes: Wednesday April 25, respectively on Tuesday, April 24, 2018.

1. Solve the heat equation \( f_t = 2018 f_{xx} \) on \([0, \pi]\) with the initial condition \( f(x, 0) = 0 \) if \( x \in [0, \pi/2] \) and \( f(x) = \sin(2x) \) if \( x \in [\pi/2, \pi] \).

2. Solve the partial differential equation \( u_t = 3u_{xxxx} + 5u_{xx} \) with initial condition \( u(0, x) = 21x \).

3. A piano string is fixed at the ends \( x = 0 \) and \( x = \pi \) and is initially undisturbed \( u(x, 0) = 0 \). The piano hammer induces an initial velocity \( u_t(x, 0) = g(x) \) onto the string, where \( g(x) = \sin(3x) \) on the interval \([0, \pi/2]\) and \( g(x) = 0 \) on \([\pi/2, \pi]\). How does the string amplitude \( u(x, t) \) move, if it follows the wave equation \( u_{tt} = u_{xx} \)?

4. A laundry line is excited by the wind. It satisfies the differential equation \( u_{tt} = u_{xx} + \cos(t) + \cos(3t) \). Assume that the amplitude \( u \) satisfies initial position \( u(x, 0) = x \) and \( u_t(x, 0) = 4 \sin(5x) + 10 \sin(6x) \). Find the function \( u(x, t) \) which satisfies the differential equation.

   Hint. First find the general homogeneous solution \( u_{\text{homogeneous}} \) of \( u_{tt} = u_{xx} \) for an odd \( u \) then a particular solution \( u_{\text{particular}}(t) \) which only depends on \( t \). Finally fix the Fourier coefficients.

5. We have looked at 4 different types of differential equations. Systems of linear differential equations \( x' = Ax \), nonlinear equations \( x' = f(x, y), y' = g(x, y) \), inhomogeneous equations \( p(D)f = g \) as well as partial differential equations like \( u_t = D^2u \) and the wave equation \( u_{tt} = D^2u \). Give an original example of each type (it should not have appeared in any homework nor handout).
Solving a PDE means to find a function $f$ on the interval $[0, \pi]$. We write it as a sin-series which means that we only need to compute the $b_n$ using the formula

$$\frac{2}{\pi} \int_0^\pi f(x) \sin(n x) \, dx.$$ 

This is justified as we can think of $f$ continued as $f(-x) = -f(x)$ on $[-\pi, 0]$ The temperature distribution $f(x, t)$ in a metal bar $[0, \pi]$ satisfies the **heat equation**

$$f_t(x, t) = \mu f_{xx}(x, t) = D^2 f(x, t)$$

Here $\mu$ is a positive constant which depends on the material. The height of a string $f(x, t)$ at time $t$ and position $x$ on $[0, \pi]$ satisfies the **wave equation**

$$f_{tt}(x, t) = c^2 f_{xx}(x, t) = c^2 D^2(f)(x, t)$$

Here $c$ is a positive constant which tell how fast the waves move. All problems are solved by diagonalizing $D^2$ using a Fourier basis. For heat, write the initial condition as a Fourier series and write down the solution. For wave, write both the initial condition $f(0, x)$ as well as the initial velocity $f_t(0, x)$ as a Fourier series and write down the solutions.