Sorting and Algorithm Analysis

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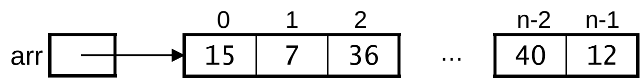
Sorting an Array of Integers



- Ground rules:
  - sort the values in increasing order
  - sort “in place,” using only a small amount of additional storage
- Terminology:
  - position: one of the memory locations in the array
  - element: one of the data items stored in the array
  - element i: the element at position i
- Goal: minimize the number of **comparisons *C*** and the number of **moves *M*** needed to sort the array.
  - move = copying an element from one position to another
    example: `arr[3] = arr[5];`

Defining a Class for our Sort Methods

```
public class Sort {
    public static void bubbleSort(int[] arr) {
        ...
    }
    public static void insertionSort(int[] arr) {
        ...
    }
    ...
}
```

- Our `Sort` class is simply a collection of methods like Java's built-in `Math` class.
- Because we never create `Sort` objects, all of the methods in the class must be *static*.
  - outside the class, we invoke them using the class name: e.g., `Sort.bubbleSort(arr)`.
- `~cscie119/examples/sorting/Sort.java`

Defining a Swap Method

- It would be helpful to have a method that swaps two elements of the array.
- Why won't the following work?

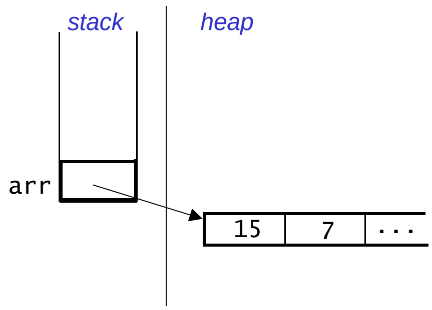
```
public static void swap(int a, int b) {
    int temp = a;
    a = b;
    b = temp;
}
```

An Incorrect Swap Method

```
public static void swap(int a, int b) {
    int temp = a;
    a = b;
    b = temp;
}
```

- Trace through the following lines to see the problem:

```
int[] arr = {15, 7, …};
swap(arr[0], arr[1]);
```



A Correct Swap Method

- This method works:

```
public static void swap(int[] arr, int a, int b) {
    int temp = arr[a];
    arr[a] = arr[b];
    arr[b] = temp;
}
```

- Trace through the following with a memory diagram to convince yourself that it works:

```
int[] arr = {15, 7, …};
swap(arr, 0, 1);
```

Selection Sort

- Basic idea:
  - consider the positions in the array from left to right
  - for each position, find the element that belongs there and put it in place by swapping it with the element that's currently there
- Example:

0	1	2	3	4
15	6	*2*	12	4

0	*1*	2	3	4
2	6	15	12	*4*

0	1	*2*	3	4
2	4	15	12	*6*

0	1	2	*3*	4
2	4	6	*12*	15

Why don't we need to consider position 4?

Selecting an Element

- When we consider position `i`, the elements in positions 0 through `i` – 1 are already in their final positions.

example for `i` = 3:

0	1	2	3	4	5	6
2	4	7	21	25	10	17

- To select an element for position `i`:
  - consider elements `i`, `i+1`, `i+2`, …, `arr.length` – 1, and keep track of `indexMin`, the index of the smallest element seen thus far

`indexMin: 3, 5`

0	1	2	3	4	5	6
2	4	7	21	25	*10*	17

  - when we finish this pass, `indexMin` is the index of the element that belongs in position `i`.
  - swap `arr[i]` and `arr[indexMin]`:

0	1	2	3	4	5	6
2	4	7	10	25	21	17

Implementation of Selection Sort

- Use a helper method to find the index of the smallest element:

```
private static int indexSmallest(int[] arr,
  int lower, int upper) {
    int indexMin = lower;

    for (int i = lower+1; i <= upper; i++)
        if (arr[i] < arr[indexMin])
            indexMin = i;

    return indexMin;
}
```

- The actual sort method is very simple:

```
public static void selectionSort(int[] arr) {
    for (int i = 0; i < arr.length-1; i++) {
        int j = indexSmallest(arr, i, arr.length-1);
        swap(arr, i, j);
    }
}
```

Time Analysis

- Some algorithms are much more efficient than others.
- The *time efficiency* or *time complexity* of an algorithm is some measure of the number of “operations” that it performs.
  - for sorting algorithms, we’ll focus on two types of operations: comparisons and moves
- The number of operations that an algorithm performs typically depends on the size, n, of its input.
  - for sorting algorithms, n is the # of elements in the array
  - `C(n)` = number of comparisons
  - `M(n)` = number of moves
- To express the time complexity of an algorithm, we’ll express the number of operations performed as a function of `n`.
  - examples: $C(n) = n^2 + 3n$
    $M(n) = 2n^2 - 1$

Counting Comparisons by Selection Sort

```
private static int indexSmallest(int[] arr, int lower, int upper){
    int indexMin = lower;

    for (int i = lower+1; i <= upper; i++)
        if (arr[i] < arr[indexMin])
            indexMin = i;

    return indexMin;
}
public static void selectionSort(int[] arr) {
    for (int i = 0; i < arr.length-1; i++) {
        int j = indexSmallest(arr, i, arr.length-1);
        swap(arr, i, j);
    }
}
```

- To sort n elements, selection sort performs n - 1 passes:

  on 1st pass, it performs n - 1 comparisons to find `indexSmallest`
  on 2nd pass, it performs n - 2 comparisons
  ...
  on the (n-1)st pass, it performs 1 comparison

- Adding up the comparisons for each pass, we get:

  $C(n) = 1 + 2 + \ldots + (n - 2) + (n - 1)$

Counting Comparisons by Selection Sort (cont.)

- The resulting formula for $C(n)$ is the sum of an arithmetic sequence:

$$C(n) = 1 + 2 + \ldots + (n - 2) + (n - 1) = \sum_{i=1}^{n-1} i$$

- Formula for the sum of this type of arithmetic sequence:

$$\sum_{i=1}^{m} i = \frac{m(m+1)}{2}$$

- Thus, we can simplify our expression for C(n) as follows:

$$C(n) = \sum_{i=1}^{n-1} i$$

$$= \frac{(n-1)((n-1)+1)}{2}$$

$$= \frac{(n-1)n}{2}$$

$$\boxed{C(n) = n^2/2 - n/2}$$

Focusing on the Largest Term

- When n is large, mathematical expressions of n are dominated by their "largest" term — i.e., the term that grows fastest as a function of n.
  - example:

n	$n^2/2$	$n/2$	$n^2/2 - n/2$
10	50	5	45
100	5000	50	4950
10000	50,000,000	5000	49,995,000

- In characterizing the time complexity of an algorithm, we'll focus on the largest term in its operation-count expression.
  - for selection sort, $C(n) = n^2/2 - n/2 \approx n^2/2$
- In addition, we'll typically ignore the coefficient of the largest term (e.g., $n^2/2 \rightarrow n^2$).

Big-*O* Notation

- We specify the largest term using big-*O* notation.
  - e.g., we say that $C(n) = n^2/2 - n/2$ is $O(n^2)$
- Common classes of algorithms:

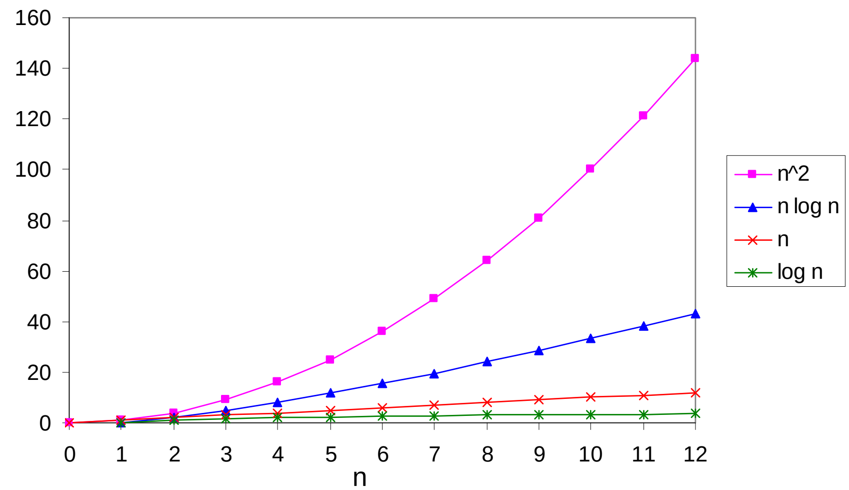
name	example expressions	big-O notation
constant time	$1, 7, 10$	$O(1)$
logarithmic time	$3\log_{10}n, \log_2 n + 5$	$O(\log n)$
linear time	$5n, 10n - 2\log_2 n$	$O(n)$
$n\log n$ time	$4n\log_2 n, n\log_2 n + n$	$O(n\log n)$
quadratic time	$2n^2 + 3n, n^2 - 1$	$O(n^2)$
exponential time	$2^n, 5e^n + 2n^2$	$O(c^n)$

slower

- For large inputs, efficiency matters more than CPU speed.
  - e.g., an $O(\log n)$ algorithm on a slow machine will outperform an $O(n)$ algorithm on a fast machine

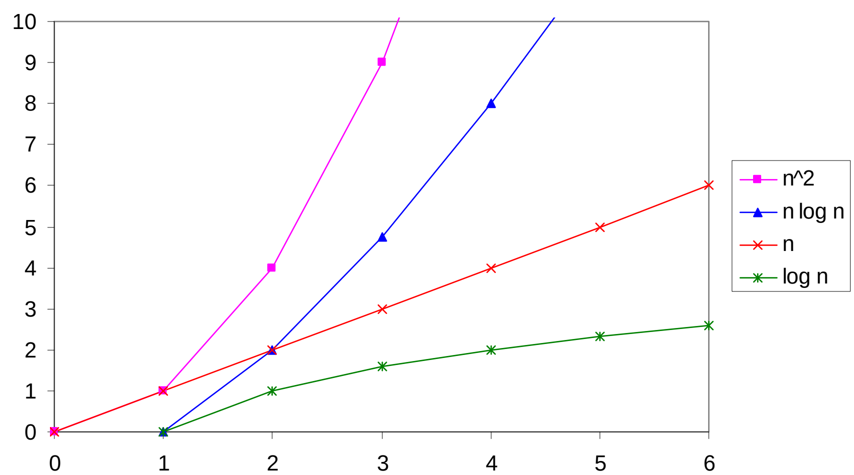
Ordering of Functions

- We can see below that: n^2 grows faster than $n\log_2 n$
  $n\log_2 n$ grows faster than n
  n grows faster than $\log_2 n$



Ordering of Functions (cont.)

- Zooming in, we see that: $n^2 >= n$ for all $n >= 1$
  $n\log_2 n >= n$ for all $n >= 2$
  $n > \log_2 n$ for all $n >= 1$

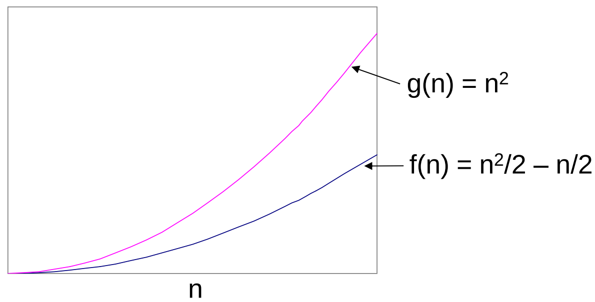


Mathematical Definition of Big-*O* Notation

- $f(n) = O(g(n))$ if there exist positive constants c and n_0 such that $f(n) <= c\,g(n)$ for all $n >= n_0$
- Example: $f(n) = n^2/2 – n/2$ is $O(n^2)$, because

  $n^2/2 – n/2 <= n^2$ for all $n >= 0$.

  $c = 1$ $n_0 = 0$



- Big-*O* notation specifies an *upper bound* on a function f(n) as n grows large.

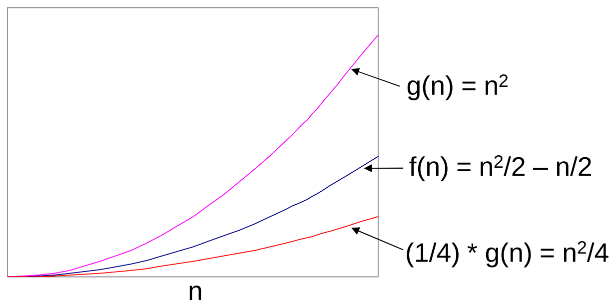
Big-*O* Notation and Tight Bounds

- Big-O notation provides an upper bound, *not* a tight bound (upper and lower).
- Example:
  - $3n – 3$ is $O(n^2)$ because $3n – 3 <= n^2$ for all $n >= 1$
  - $3n – 3$ is also $O(2^n)$ because $3n – 3 <= 2^n$ for all $n >= 1$
- However, we generally try to use big-O notation to characterize a function as closely as possible – i.e., as if we were using it to specify a tight bound.
  - for our example, we would say that $3n – 3$ is $O(n)$

Big-Theta Notation

- In theoretical computer science, *big-theta* notation (Θ) is used to specify a tight bound.
- $f(n) = \Theta(g(n))$ if there exist constants c_1, c_2, and n_0 such that $c_1 g(n) \ <= \ f(n) \ <= \ c_2 g(n)$ for all $n > n_0$
- Example: $f(n) = n^2/2 - n/2$ is $\Theta(n^2)$, because
  $(1/4)*n^2 <= n^2/2 - n/2 <= n^2$ for all $n >= 2$

$c_1 = 1/4$ $c_2 = 1$ $n_0 = 2$



Big-*O* Time Analysis of Selection Sort

- Comparisons: we showed that `C(n)` = $n^2/2 - n/2$
  - selection sort performs $O(n^2)$ comparisons
- Moves: after each of the `n-1` passes to find the smallest remaining element, the algorithm performs a swap to put the element in place.
  - `n-1` swaps, 3 moves per swap
  - `M(n) = 3(n-1) = 3n-3`
  - selection sort performs $O(n)$ moves.
- Running time (i.e., total operations): ?

Sorting by Insertion I: Insertion Sort

- Basic idea:
  - going from left to right, “insert” each element into its proper place with respect to the elements to its left, “sliding over” other elements to make room.
- Example:

0	1	2	3	4
15	*4*	2	12	6
4	15	*2*	12	6
2	4	15	*12*	6
2	4	12	15	*6*
2	4	6	12	15

Comparing Selection and Insertion Strategies

- In selection sort, we start with the *positions* in the array and *select* the correct elements to fill them.
- In insertion sort, we start with the *elements* and determine where to *insert* them in the array.
- Here’s an example that illustrates the difference:

0	1	2	3	4	5	6
18	12	15	9	25	2	17

- Sorting by selection:
  - consider position 0: find the element (2) that belongs there
  - consider position 1: find the element (9) that belongs there
  - …
- Sorting by insertion:
  - consider the 12: determine where to insert it
  - consider the 15; determine where to insert it
  - …

Inserting an Element

- When we consider element `i`, elements 0 through `i – 1` are already sorted with respect to each other.

example for `i = 3`:

0	1	2	3	4
6	14	19	9	...

- To insert element `i`:
  - make a copy of element `i`, storing it in the variable `toInsert`:

`toInsert` 9

0	1	2	3
6	14	19	9

  - consider elements `i-1`, `i-2`, …
    - if an element `>` `toInsert`, slide it over to the right
    - stop at the first element `<=` `toInsert`

`toInsert` 9

0	1	2	3
6		14	19

  - copy `toInsert` into the resulting "hole":

0	1	2	3
6	9	14	19

Insertion Sort Example (done together)

description of steps

12	5	2	13	18	4

Implementation of Insertion Sort

```
public class Sort {
  ...
  public static void insertionSort(int[] arr) {
      for (int i = 1; i < arr.length; i++) {
          if (arr[i] < arr[i-1]) {
              int toInsert = arr[i];

              int j = i;
              do {
                  arr[j] = arr[j-1];
                  j = j - 1;
              } while (j > 0  &&  toInsert < arr[j-1]);

              arr[j] = toInsert;
          }
      }
  }
}
```

Time Analysis of Insertion Sort

- The number of operations depends on the contents of the array.
- *best case:*
- *worst case:*
- *average case:*

Sorting by Insertion II: Shell Sort

- Developed by Donald Shell in 1959
- Improves on insertion sort
- Takes advantage of the fact that insertion sort is fast when an array is almost sorted.
- Seeks to eliminate a disadvantage of insertion sort: if an element is far from its final location, many "small" moves are required to put it where it belongs.
- Example: if the largest element starts out at the beginning of the array, it moves one place to the right on *every* insertion!

0	1	2	3	4	5	…	1000
999	42	56	30	18	23	…	11

- Shell sort uses "larger" moves that allow elements to quickly get close to where they belong.

Sorting Subarrays

- Basic idea:
  - use insertion sort on subarrays that contain elements separated by some increment
    - increments allow the data items to make larger "jumps"
  - repeat using a decreasing sequence of increments
- Example for an initial increment of 3:

0	1	2	3	4	5	6	7
36	18	10	27	3	20	9	8

  - three subarrays:
    1) elements 0, 3, 6   2) elements 1, 4, 7   3) elements 2 and 5
- Sort the subarrays using insertion sort to get the following:

0	1	2	3	4	5	6	7
9	3	10	27	8	20	36	18

- Next, we complete the process using an increment of 1.

Shell Sort: A Single Pass

- We *don't* consider the subarrays one at a time.
- We consider elements `arr[incr]` through `arr[arr.length-1]`, inserting each element into its proper place with respect to the elements *from its subarray* that are to the left of the element.
- The same example (`incr` = 3):

0	1	2	3	4	5	6	7
36	18	10	*27*	3	20	9	8
27	*18*	10	36	*3*	20	9	8
27	3	*10*	36	18	*20*	9	8
27	3	10	*36*	18	20	*9*	8
9	*3*	10	27	*18*	20	36	*8*
9	3	10	27	8	20	36	18

Inserting an Element in a Subarray

- When we consider element `i`, the other elements in its subarray are already sorted with respect to each other.

example for `i = 6`:
(`incr = 3`)

0	1	2	3	4	5	6	7
27	3	10	*36*	18	20	*9*	8

the other element's in 9's subarray (the 27 and 36)
are already sorted with respect to each other

- To insert element `i`:
  - make a copy of element `i`, storing it in the variable `toInsert`:

`toInsert` | *9* |

0	1	2	3	4	5	6	7
27	3	10	*36*	18	20	*9*	8

  - consider elements `i-incr`, `i-(2*incr)`, `i-(3*incr)`,...
    - if an element `>` `toInsert`, slide it right *within the subarray*
    - stop at the first element `<=` `toInsert`

`toInsert` | *9* |

0	1	2	3	4	5	6	7
	3	10	*27*	18	20	*36*	8

  - copy `toInsert` into the "hole":

0	1	2	3	4	
9	3	10	*27*	18	...

The Sequence of Increments

- Different sequences of decreasing increments can be used.
- Our version uses values that are one less than a power of two.
  - $2^k - 1$ for some k
  - … 63, 31, 15, 7, 3, 1
  - can get to the next lower increment using integer division:

    ```
    incr = incr/2;
    ```
- Should avoid numbers that are multiples of each other.
  - otherwise, elements that are sorted with respect to each other in one pass are grouped together again in subsequent passes
    - repeat comparisons unnecessarily
    - get fewer of the large jumps that speed up later passes
  - example of a bad sequence: 64, 32, 16, 8, 4, 2, 1
    - what happens if the largest values are all in odd positions?

Implementation of Shell Sort

```
public static void shellSort(int[] arr) {
    int incr = 1;
    while (2 * incr <= arr.length)
        incr = 2 * incr;
    incr = incr - 1;

    while (incr >= 1) {
        for (int i = incr; i < arr.length; i++) {
            if (arr[i] < arr[i-incr]) {
                int toInsert = arr[i];

                int j = i;
                do {
                    arr[j] = arr[j-incr];
                    j = j - incr;
                } while (j > incr-1 &&
                   toInsert < arr[j-incr]);

                arr[j] = toInsert;
            }
        }
        incr = incr/2;
    }
}
```

(If you replace `incr` with `1` in the for-loop, you get the code for insertion sort.)

Time Analysis of Shell Sort

- Difficult to analyze precisely
  - typically use experiments to measure its efficiency
- With a bad interval sequence, it's $O(n^2)$ in the worst case.
- With a good interval sequence, it's better than $O(n^2)$.
  - at least $O(n^{1.5})$ in the average and worst case
  - some experiments have shown average-case running times of $O(n^{1.25})$ or even $O(n^{7/6})$
- Significantly better than insertion or selection for large n:

n	n^2	$n^{1.5}$	$n^{1.25}$
10	100	31.6	17.8
100	10,000	1000	316
10,000	100,000,000	1,000,000	100,000
10^6	10^{12}	10^9	3.16×10^7

- We've wrapped insertion sort in another loop and increased its efficiency! The key is in the larger jumps that Shell sort allows.

Sorting by Exchange I: Bubble Sort

- Perform a sequence of passes through the array.
- On each pass: proceed from left to right, swapping adjacent elements if they are out of order.
- Larger elements "bubble up" to the end of the array.
- At the end of the kth pass, the k rightmost elements are in their final positions, so we don't need to consider them in subsequent passes.
- Example:

	0	1	2	3
	28	24	27	18
after the first pass:	24	27	18	*28*
after the second:	24	18	*27*	*28*
after the third:	18	*24*	*27*	*28*

Implementation of Bubble Sort

```
public class Sort {
    ...
    public static void bubbleSort(int[] arr) {
        for (int i = arr.length - 1; i > 0; i--) {
            for (int j = 0; j < i; j++) {
                if (arr[j] > arr[j+1])
                    swap(arr, j, j+1);
            }
        }
    }
}
```

- One for-loop nested in another:
    - the inner loop performs a single pass
    - the outer loop governs the number of passes, and the ending point of each pass

Time Analysis of Bubble Sort

- Comparisons: the kth pass performs ______ comparisons,

  so we get `C(n) =`

- Moves: depends on the contents of the array
    - in the worst case:
    - in the best case:
- Running time:

Sorting by Exchange II: Quicksort

- Like bubble sort, quicksort uses an approach based on exchanging out-of-order elements, but it's more efficient.
- A recursive, divide-and-conquer algorithm:
  - *divide:* rearrange the elements so that we end up with two subarrays that meet the following criterion:

    each element in the left array <= each element in the right array

    example:

12	8	14	4	6	13

→

6	8	4	14	12	13

  - *conquer:* apply quicksort recursively to the subarrays, stopping when a subarray has a single element
  - *combine:* nothing needs to be done, because of the criterion used in forming the subarrays

Partitioning an Array Using a Pivot

- The process that quicksort uses to rearrange the elements is known as *partitioning* the array.
- Partitioning is done using a value known as the *pivot.*
- We rearrange the elements to produce two subarrays:
  - left subarray: all values <= pivot
  - right subarray: all values >= pivot

  equivalent to the criterion on the previous page.

7	15	4	9	6	18	9	12

↓ *partition using a pivot of 9*

7	9	4	6	9	18	15	12

all values <= 9 | all values >= 9

- Our approach to partitioning is one of several variants.
- Partitioning is useful in its own right.
  ex: find all students with a GPA > 3.0.

Possible Pivot Values

- First element or last element
  - risky, can lead to terrible worst-case behavior
  - especially poor if the array is almost sorted

4	8	14	12	6	18

pivot = 18

→

4	8	14	12	6	18

- Middle element (what we will use)
- Randomly chosen element
- Median of three elements
  - left, center, and right elements
  - three randomly selected elements
  - taking the median of three decreases the probability of getting a poor pivot

Partitioning an Array: An Example

`arr` →

first							last
7	15	4	*9*	6	18	9	12

`pivot = 9`

- Maintain indices `i` and `j`, starting them "outside" the array:

`i = first - 1`
`j = last + 1`

i									j
	7	15	4	9	6	18	9	12	

- *Find* "out of place" elements:
  - increment i until `arr[i] >= pivot`
  - decrement j until `arr[j] <= pivot`

	i					j	
7	15	4	9	6	18	9	12

- *Swap* `arr[i]` and `arr[j]`:

	i					j	
7	9	4	9	6	18	15	12

Partitioning Example (cont.)

from prev. page:

	`i`					`j`	
7	9	4	9	6	18	15	12

- Find:

			`i`	`j`			
7	9	4	9	6	18	15	12

- Swap:

			`i`	`j`			
7	9	4	6	9	18	15	12

- Find:

			`j`	`i`			
7	9	4	6	9	18	15	12

  and now the indices have crossed, so we return `j`.

- Subarrays: left = `arr[first:j]`, right = `arr[j+1:last]`

`first`			`j`	`i`			`last`
7	9	4	6	9	18	15	12

Partitioning Example 2

- Start (pivot = 13):

`i`									`j`
	24	5	2	*13*	18	4	20	19	

- Find:

`i`					`j`		
24	5	2	13	18	4	20	19

- Swap:

`i`					`j`		
4	5	2	13	18	24	20	19

- Find:

			`i j`				
4	5	2	13	18	24	20	19

  and now the indices are equal, so we return `j`.

- Subarrays:

			`i j`				
4	5	2	13	18	24	20	19

Partitioning Example 3 (done together)

- Start (pivot = 5):

i								j
	4	14	7	*5*	2	19	26	6

- Find:

4	14	7	5	2	19	26	6

partition() Helper Method

```
private static int partition(int[] arr, int first, int last)
{
    int pivot = arr[(first + last)/2];
    int i = first - 1;  // index going left to right
    int j = last + 1;   // index going right to left
    while (true) {
        do {
            i++;
        } while (arr[i] < pivot);
        do {
            j--;
        } while (arr[j] > pivot);
        if (i < j)
            swap(arr, i, j);
        else
            return j;   // arr[j] = end of left array
    }
}
```

Implementation of Quicksort

```
public static void quickSort(int[] arr) {
    qSort(arr, 0, arr.length - 1);
}

private static void qSort(int[] arr, int first, int last) {
    int split = partition(arr, first, last);

    if (first < split)
        qSort(arr, first, split);      // left subarray
    if (last > split + 1)
        qSort(arr, split + 1, last);   // right subarray
}
```

Counting Students: Divide and Conquer

- Everyone stand up.
- You will each carry out the following algorithm:

```
count = 1;

while (you are not the only person standing) {
    find another person who is standing
    if (your first name < other person's first name)
        sit down (break ties using last names)
    else
        count = count + the other person's count
}

if (you are the last person standing)
    report your final count
```

Counting Students: Divide and Conquer (cont.)

- At each stage of the "joint algorithm", the problem size is divided in half.

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☺

- How many stages are there as a function of the number of students, n?
- This approach benefits from the fact that you perform the algorithm *in parallel* with each other.

A Quick Review of Logarithms

- $\log_b n$ = the exponent to which b must be raised to get n
  - $\log_b n = p$ if $b^p = n$
  - examples: $\log_2 8 = 3$ because $2^3 = 8$
    $\log_{10} 10000 = 4$ because $10^4 = 10000$
- Another way of looking at logs:
  - let's say that you repeatedly divide n by b (using integer division)
  - $\log_b n$ is an upper bound on the number of divisions needed to reach 1
  - example: $\log_2 18$ is approx. 4.17
    18/2 = 9   9/2 = 4   4/2 = 2   2/2 = 1

A Quick Review of Logs (cont.)

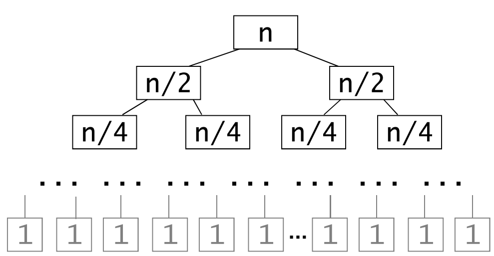
- If the number of operations performed by an algorithm is proportional to $\log_b n$ for any base b, we say it is a $O(\log n)$ algorithm – dropping the base.
- $\log_b n$ grows much more slowly than n

n	$\log_2 n$
2	1
1024 (1K)	10
1024*1024 (1M)	20

- Thus, for large values of n:
  - a $O(\log n)$ algorithm is much faster than a $O(n)$ algorithm
  - a $O(n \log n)$ algorithm is much faster than a $O(n^2)$ algorithm
- We can also show that an $O(n \log n)$ algorithm is faster than a $O(n^{1.5})$ algorithm like Shell sort.

Time Analysis of Quicksort

- Partitioning an array requires n comparisons, because each element is compared with the pivot.
- *best case:* partitioning always divides the array in half
  - repeated recursive calls give:



<u>*comparisons*</u>

n

2*(n/2) = n

4*(n/4) = n

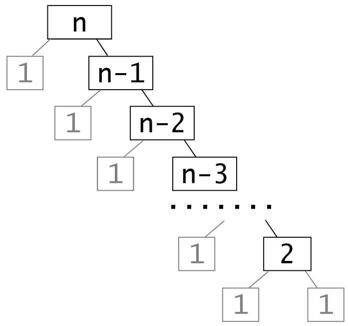
...

0

  - at each "row" except the bottom, we perform n comparisons
  - there are ________ rows that include comparisons
  - C(n) = ?
  - Similarly, M(n) and running time are both ___________

Time Analysis of Quicksort (cont.)

- *worst case:* pivot is always the smallest or largest element
  - one subarray has 1 element, the other has n - 1
  - repeated recursive calls give:



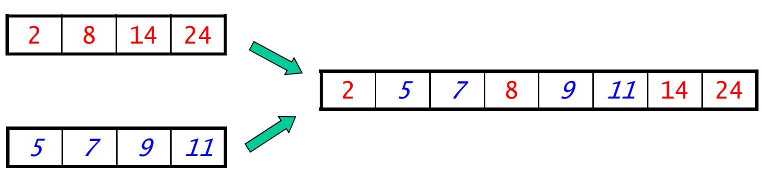
comparisons
n
n-1
n-2
n-3
...
2

  - $C(n) = \sum_{i=2}^{n} i = O(n^2)$. M(n) and run time are also $O(n^2)$.

- *average case* is harder to analyze
  - $C(n) > n \log_2 n$, but it's still $O(n \log n)$

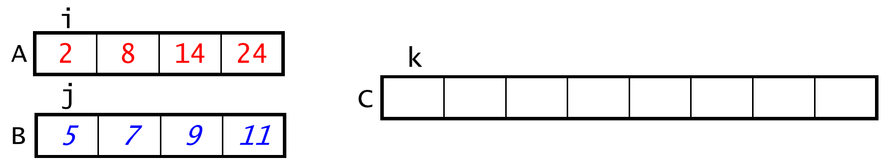
Mergesort

- All of the comparison-based sorting algorithms that we've seen thus far have sorted the array in place.
  - used only a small amount of additional memory
- Mergesort is a sorting algorithm that requires an additional temporary array of the same size as the original one.
  - it needs $O(n)$ additional space, where n is the array size
- It is based on the process of *merging* two sorted arrays into a single sorted array.
  - example:

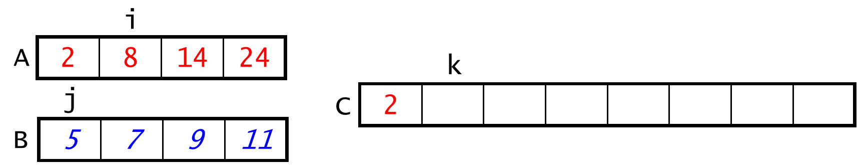


Merging Sorted Arrays

- To merge sorted arrays A and B into an array C, we maintain three indices, which start out on the first elements of the arrays:

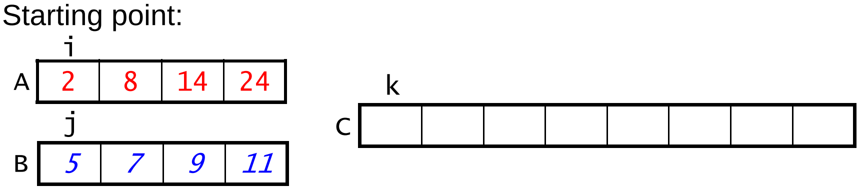


- We repeatedly do the following:
  - compare A[i] and B[j]
  - copy the smaller of the two to C[k]
  - increment the index of the array whose element was copied
  - increment k

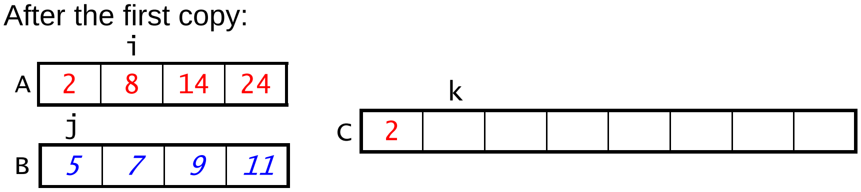


Merging Sorted Arrays (cont.)

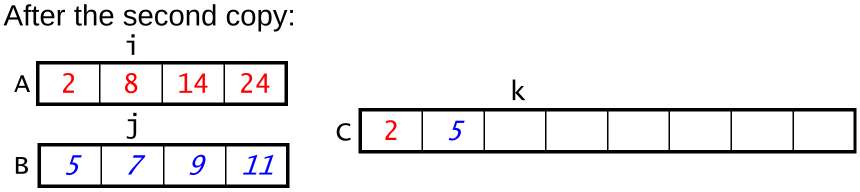
- Starting point:



- After the first copy:

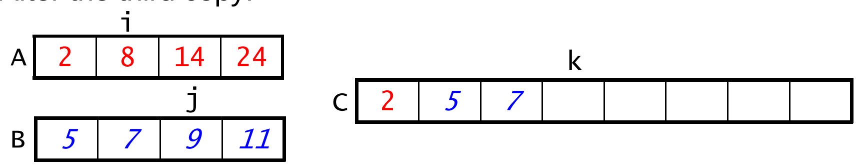


- After the second copy:

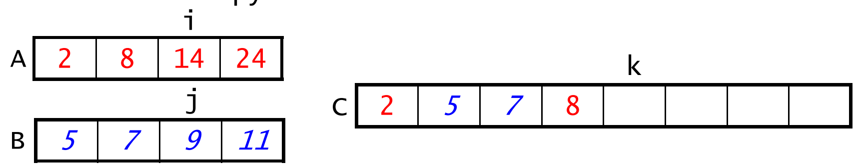


Merging Sorted Arrays (cont.)

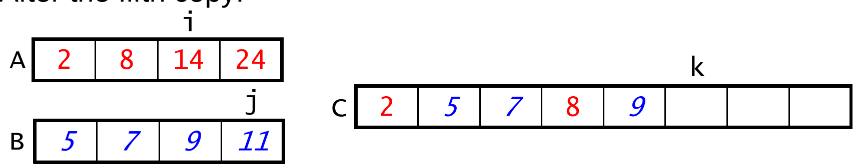
- After the third copy:



- After the fourth copy:

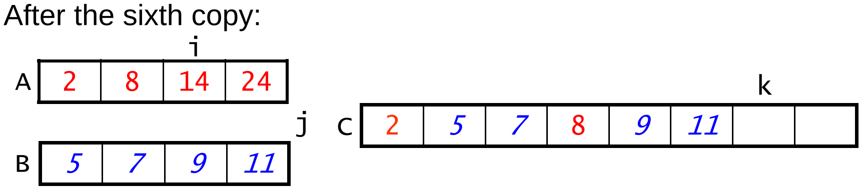


- After the fifth copy:

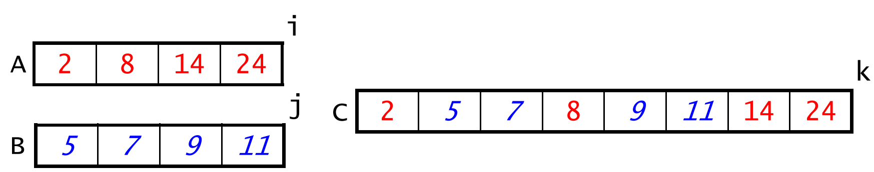


Merging Sorted Arrays (cont.)

- After the sixth copy:

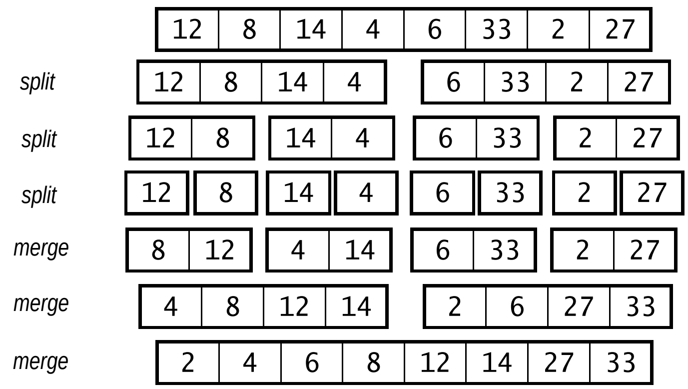


- There's nothing left in B, so we simply copy the remaining elements from A:



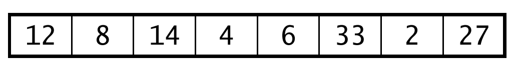
Divide and Conquer

- Like quicksort, mergesort is a divide-and-conquer algorithm.
  - *divide:* split the array in half, forming two subarrays
  - *conquer:* apply mergesort recursively to the subarrays, stopping when a subarray has a single element
  - *combine:* merge the sorted subarrays

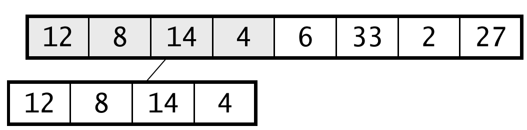


Tracing the Calls to Mergesort

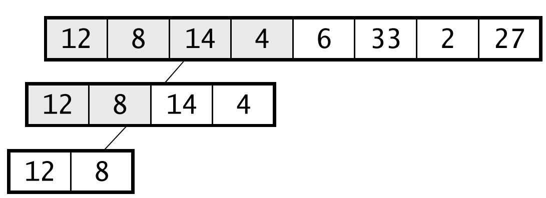
the initial call is made to sort the entire array:



split into two 4-element subarrays, and make a recursive call to sort the left subarray:

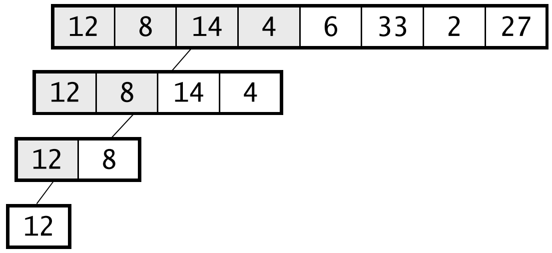


split into two 2-element subarrays, and make a recursive call to sort the left subarray:

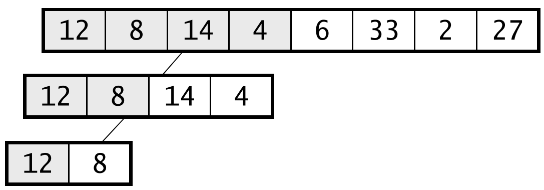


Tracing the Calls to Mergesort

split into two 1-element subarrays, and make a recursive call to sort the left subarray:

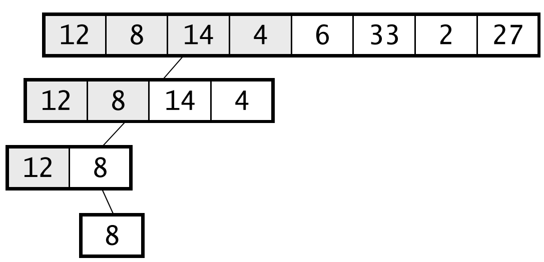


base case, so return to the call for the subarray {12, 8}:

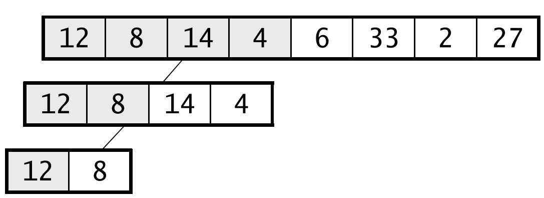


Tracing the Calls to Mergesort

make a recursive call to sort its right subarray:

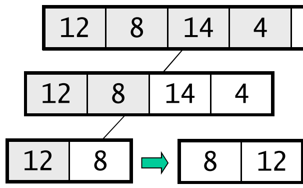


base case, so return to the call for the subarray {12, 8}:

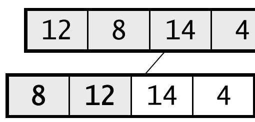


Tracing the Calls to Mergesort

merge the sorted halves of {12, 8}:

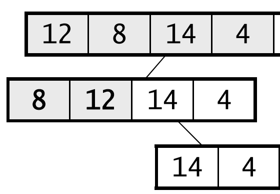


end of the method, so return to the call for the 4-element subarray, which now has a sorted left subarray:

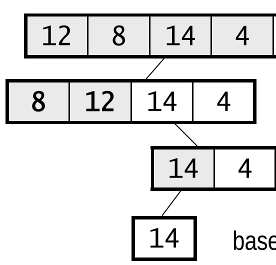


Tracing the Calls to Mergesort

make a recursive call to sort the right subarray of the 4-element subarray



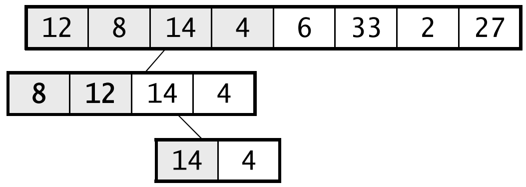
split it into two 1-element subarrays, and make a recursive call to sort the left subarray:



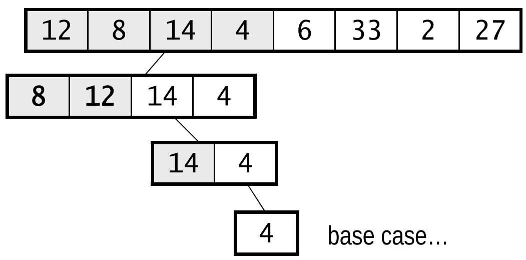
base case…

Tracing the Calls to Mergesort

return to the call for the subarray {14, 4}:



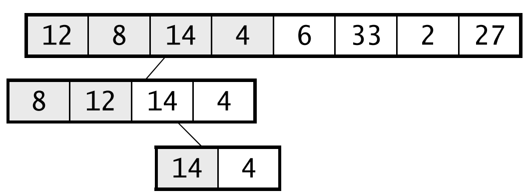
make a recursive call to sort its right subarray:



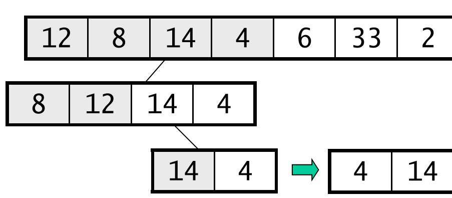
base case…

Tracing the Calls to Mergesort

return to the call for the subarray {14, 4}:



merge the sorted halves of {14, 4}:



Tracing the Calls to Mergesort

end of the method, so return to the call for the 4-element subarray, which now has two sorted 2-element subarrays:

12	8	14	4	6	33	2	27

8	12	4	14

merge the 2-element subarrays:

12	8	14	4	6	33	2	27

8	12	4	14	⇨	4	8	12	14

Tracing the Calls to Mergesort

end of the method, so return to the call for the original array, which now has a sorted left subarray:

4	**8**	**12**	**14**	6	33	2	27

perform a similar set of recursive calls to sort the right subarray. here's the result:

4	8	12	14	**2**	**6**	**27**	**33**

finally, merge the sorted 4-element subarrays to get a fully sorted 8-element array:

4	8	12	14	2	6	27	33

⇩

2	4	6	8	12	14	27	33

Implementing Mergesort

- One approach is to create new arrays for each new set of subarrays, and to merge them back into the array that was split.
- Instead, we'll create a temp. array of the same size as the original.
  - pass it to each call of the recursive mergesort method
  - use it when merging subarrays of the original array:

arr	8	12	4	14	6	33	2	27
temp	4	8	12	14				

  - after each merge, copy the result back into the original array:

arr	4	8	12	14	6	33	2	27
temp	4	8	12	14				

A Method for Merging Subarrays

```
public static void merge(int[] arr, int[] temp,
  int leftStart, int leftEnd, int rightStart, int rightEnd) {
    int i = leftStart;    // index into left subarray
    int j = rightStart;   // index into right subarray
    int k = leftStart;    // index into temp

    while (i <= leftEnd && j <= rightEnd) {
        if (arr[i] < arr[j])
            temp[k++] = arr[i++];
        else
            temp[k++] = arr[j++];
    }

    while (i <= leftEnd)
        temp[k++] = arr[i++];

    while (j <= rightEnd)
        temp[k++] = arr[j++];

    for (i = leftStart; i <= rightEnd; i++)
        arr[i] = temp[i];
}
```

Methods for Mergesort

- We use a wrapper method to create the temp. array, and to make the initial call to a separate recursive method:

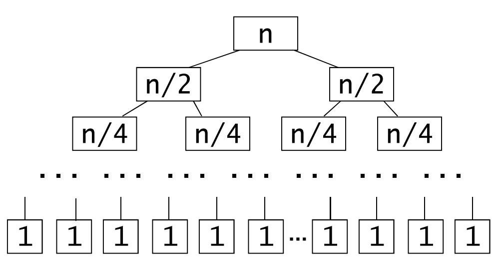
```
public static void mergeSort(int[] arr) {
    int[] temp = new int[arr.length];
    mSort(arr, temp, 0, arr.length - 1);
}
```

- Let's implement the recursive method together:

```
public static void mSort(int[] arr, int[] temp,
  int start, int end) {









}
```

Time Analysis of Mergesort

- Merging two halves of an array of size n requires 2n moves. Why?
- Mergesort repeatedly divides the array in half, so we have the following call tree (showing the sizes of the arrays):



moves

`2n`

`2*2*(n/2) = 2n`

`4*2*(n/4) = 2n`

`...`

  - at all but the last level of the call tree, there are 2n moves
  - how many levels are there?
  - `M(n)` = ?
  - `C(n)` = ?

Summary: Comparison-Based Sorting Algorithms

algorithm	best case	avg case	worst case	extra memory
selection sort	$O(n^2)$	$O(n^2)$	$O(n^2)$	$O(1)$
insertion sort	$O(n)$	$O(n^2)$	$O(n^2)$	$O(1)$
Shell sort	$O(n \log n)$	$O(n^{1.5})$	$O(n^{1.5})$	$O(1)$
bubble sort	$O(n^2)$	$O(n^2)$	$O(n^2)$	$O(1)$
quicksort	$O(n \log n)$	$O(n \log n)$	$O(n^2)$	$O(1)$
mergesort	$O(n \log n)$	$O(n \log n)$	$O(n \log n)$	$O(n)$

- Insertion sort is best for nearly sorted arrays.
- Mergesort has the best worst-case complexity, but requires extra memory – and moves to and from the temp array.
- Quicksort is comparable to mergesort in the average case. With a reasonable pivot choice, its worst case is seldom seen.
- Use `~cscie119/examples/sorting/SortCount.java` to experiment.

Comparison-Based vs. Distributive Sorting

- Until now, all of the sorting algorithms we have considered have been *comparison-based:*
  - treat the keys as wholes (comparing them)
  - don't "take them apart" in any way
  - all that matters is the relative order of the keys, not their actual values.
- No comparison-based sorting algorithm can do better than $O(n \log_2 n)$ on an array of length n.
  - $O(n \log_2 n)$ is a *lower bound* for such algorithms.
- *Distributive* sorting algorithms do more than compare keys; they perform calculations on the actual values of individual keys.
- Moving beyond comparisons allows us to overcome the lower bound.
  - tradeoff: use more memory.

Distributive Sorting Example: Radix Sort

- Relies on the representation of the data as a sequence of **m** quantities with **k** possible values.
- Examples:

Examples	**m**	**k**
integer in range 0 ... 999	3	10
string of 15 upper-case letters	15	26
32-bit integer	32	2 (in binary)
	4	256 (as bytes)

- Strategy: Distribute according to the last element in the sequence, then concatenate the results:

  33 41 12 24 31 14 13 42 34

  get: 41 31 | 12 42 | 33 13 | 24 14 34

- Repeat, moving back one digit each time:

  get: | | |

Analysis of Radix Sort

- Recall that we treat the values as a sequence of `m` quantities with `k` possible values.
- Number of operations is O(`n*m`) for an array with `n` elements
  - better than O(`n log n`) when `m < log n`
- Memory usage increases as `k` increases.
  - `k` tends to increase as `m` decreases
  - tradeoff: increased speed requires increased memory usage

Big-*O* Notation Revisited

- We've seen that we can group functions into classes by focusing on the fastest-growing term in the expression for the number of operations that they perform.
  - e.g., an algorithm that performs $n^2/2 - n/2$ operations is a $O(n^2)$-time or quadratic-time algorithm

- Common classes of algorithms:

slower

name	example expressions	big-O notation
constant time	$1, 7, 10$	$O(1)$
logarithmic time	$3\log_{10}n, \log_2 n + 5$	$O(\log n)$
linear time	$5n, 10n - 2\log_2 n$	$O(n)$
$n \log n$ time	$4n\log_2 n, n\log_2 n + n$	$O(n \log n)$
quadratic time	$2n^2 + 3n, n^2 - 1$	$O(n^2)$
cubic time	$n^2 + 3n^3, 5n^3 - 5$	$O(n^3)$
exponential time	$2^n, 5e^n + 2n^2$	$O(c^n)$
factorial time	$3n!, 5n + n!$	$O(n!)$

How Does the Number of Operations Scale?

- Let's say that we have a problem size of 1000, and we measure the number of operations performed by a given algorithm.

- If we double the problem size to 2000, how would the number of operations performed by an algorithm increase if it is:
  - $O(n)$-time
  - $O(n^2)$-time
  - $O(n^3)$-time
  - $O(\log_2 n)$-time
  - $O(2^n)$-time

How Does the Actual Running Time Scale?

- How much time is required to solve a problem of size n?
  - assume that each operation requires 1 μsec (1×10^{-6} sec)

time function	problem size (n) 10	20	30	40	50	60
n	.00001 s	.00002 s	.00003 s	.00004 s	.00005 s	.00006 s
n^2	.0001 s	.0004 s	.0009 s	.0016 s	.0025 s	.0036 s
n^5	.1 s	3.2 s	24.3 s	1.7 min	5.2 min	13.0 min
2^n	.001 s	1.0 s	17.9 min	12.7 days	35.7 yrs	36,600 yrs

- sample computations:
  - when n = 10, an n^2 algorithm performs 10^2 operations.
    $10^2 * (1 \times 10^{-6}$ sec) = .0001 sec
  - when n = 30, a 2^n algorithm performs 2^{30} operations.
    $2^{30} * (1 \times 10^{-6}$ sec) = 1073 sec = 17.9 min

What's the Largest Problem That Can Be Solved?

- What's the largest problem size n that can be solved in a given time T? (again assume 1 μsec per operation)

time function	time available (T) 1 min	1 hour	1 week	1 year
n	60,000,000	3.6×10^9	6.0×10^{11}	3.1×10^{13}
n^2	7745	60,000	777,688	5,615,692
n^5	35	81	227	500
2^n	25	31	39	44

- sample computations:
  - 1 hour = 3600 sec
    that's enough time for $3600/(1 \times 10^{-6}) = 3.6 \times 10^9$ operations
    - n^2 algorithm:
      $n^2 = 3.6 \times 10^9 \rightarrow n = (3.6 \times 10^9)^{1/2} = 60{,}000$
    - 2^n algorithm:
      $2^n = 3.6 \times 10^9 \rightarrow n = \log_2(3.6 \times 10^9) \sim= 31$