color

- we have been using r, g, b..
- why
- what is a color?
- can we get all colors this way?
- how does wavelength fit in here, what part is physics, what part is physiology
- can i use r, g, b for simulation of reflection
- how should colors be stored

different meanings of color

- neural response of cone cells: retinal color
- this is processed giving us: perceived color that we actually experience and base judgments upon.
  - note we do not have direct experiential access to retinal color
  - see constancy slides
  - ted’s image and video
- The perceived color is often associated with the object we are observing, which we might call the object color.
- we organize colors and name them as well

what can we say about color

- deep: what is the experience of color
- simple: are two experiences identical

our plan

- mostly retinal color
- well understood, and is the starting point
- we will describe bio-physically
- we will re-describe using perceptual experiments and formal definitions

light beams

- Visible light is electromagnetic radiation that falls roughly in the wavelengths $380 < \lambda < 770$, measured in nanometers.
- A pure beam $l_\lambda$ has one “unit” of light of a specific wavelength $\lambda$.
- A mixed beam $l(\lambda)$ has different amounts of various wavelengths.
  - These amounts are determined by a function $l(\cdot) : R \rightarrow R_+$,
  - The value is always non-negative since there is no “negative light”.

cones

- retinal has 3 kinds of light sensitive cone cells
  - color blind people have only 2 kinds
- called long, medium and short (after the wavelengths of light they are most sensitive to).
- three sensitivity functions $k_l(\lambda)$, $k_m(\lambda)$ and $k_s(\lambda)$.
  - describes how strongly one type of cone “responds” to pure beams of light of different wavelengths.
metamers

- these 3 numbers does not have all of the information of the \( l(\lambda) \) function.
- this suggests that there may be many physically distinct lights that give us identical retinal responses.
- this means that there are many physically distinct beams of light, with different amounts of each wavelengths, that generate the same color sensation.
- We call any two such beams metamers.

visualize

- each pure beam of light results in three cone response values on the retina,
- visualize this response as a single point in a 3D space.
- define a 3D linear space, with coordinates labeled \([L, M, S]t\)
- for a fixed \( \lambda \), we can draw the retinal response as a single vector with coordinates \([k_l(\lambda), k_m(\lambda), k_s(\lambda)]t\).

lasso curve

- As we let \( \lambda \) vary, such vectors will trace out a lasso curve in space (see demo)
- The lasso curve lies completely in the positive octant since all responses are positive.
- The curve both starts and ends at the origin since these extreme wavelengths are at the boundaries of the visible region, beyond which the responses are zero.
- The curve spends a short time on the \( S \) axis (shown with blue tinted points)
- finally comes close to the \( L \) axis (shown in red).
- curve never comes close to the \( M \) axis, as there is no light that stimulates these cones alone.
- web: imaginary-and-impossible-colors.html

color space

- \([L, M, S]t\) coordinates of the pure light beam describe the (retinal) color sensation
- We use the symbol \( \vec{c} \) to represent this sensation
- in this lasso visualization we can think of each 3D vector as potentially representing some color.
  - we are not sure which ones are really achievable (negatives for starters, are not).
- Vectors on the lasso curve are the actual colors of pure beams.
  - so too are scales thereof

mixed beams

- for summed light \( \sum_i l(i) l_{\lambda_i} \), the three responses \([L, M, S]t\) are
  \[
  L = \sum_i l(i) k_l(\lambda_i) \\
  M = \sum_i l(i) k_m(\lambda_i) \\
  S = \sum_i l(i) k_s(\lambda_i)
  \]
for a mixed beam of light \( l(\lambda) \), the three responses \([L, M, S]^t\) are

\[
\begin{align*}
L &= \int_\Omega d\lambda \ l(\lambda) \ k_l(\lambda) \\
M &= \int_\Omega d\lambda \ l(\lambda) \ k_m(\lambda) \\
S &= \int_\Omega d\lambda \ l(\lambda) \ k_s(\lambda)
\end{align*}
\]

where \( \Omega = [380..770] \).

mixed in vis

- As we look at all possible mixed beams \( l(\lambda) \), the resulting \([L, M, S]^t\) coordinates sweep out some set of vectors in 3D space.
- Since \( l(\lambda) \) can be any positive function, the swept set is comprised of all positive linear combinations of vectors on the lasso curve.
- Thus, the swept set is the convex cone over the lasso curve, which we call the color cone.
- Vectors inside the cone represent actual achievable color sensations.
- Vectors outside the cone, such as the vertical axis do not arise as the sensation from any actual light beam, whether pure or composite.
- see demo

Map of Color Space

- Scales of vectors in the cone correspond to brightness changes in our perceived color sensation, so lets normalize by scale
  - see figure
- we only draw colors in the gamut of the RGB monitor
- Colors along the boundary of the cone are vivid and are perceived as “saturated”.
- As we circle around the boundary, we move through the different “hues” of color.
- Starting from the \( L \) axis, we move along the rainbow colors from red to green to violet.
  - achievable by pure beams
- color cone’s boundary has a planar wedge (a line segment in the 2D figure).
  - The colors on this wedge are the pinks and purples.
  - They do not appear in the rainbow and can only be achieved by appropriately combining beams of red and violet.
- As we move in from the boundary towards the central region of the cone, the colors, while maintaining their hue, de-saturate, becoming pastel and eventually grayish or whitish.

adding colors

- look at the form for turning \( l(\lambda) \) to \((L, M, S)\) values (color sensations).
- if you add the lights \( l_1(\lambda) + l_2(\lambda) \), this produces \((L_1 + L_2, M_1 + M_2, S_1 + S_2)\).
- if \( l'_1(\lambda) \) also produces \((L_1, M_1, S_1)\), then \( l'_1(\lambda) + l_2(\lambda) \) also produces \((L_1 + L_2, M_1 + M_2, S_1 + S_2)\).
- so we can predict the color resulting from adding two lights just from the colors of the two individual lights.

mixing paints

- a simple model of paint is that it reflects only some fraction of each wavelength \( r(\lambda) \) of an incoming beam \( i(\lambda) \).
- so the beam to you eye is \( l(\lambda) = i(\lambda) r(\lambda) \).
- this \( l(\lambda) \) has a color (3 numbers).
we might model mixing of two paints as producing a new paint through 
\[ r(\lambda) = r_1(\lambda)r_2(\lambda) \]
the color of the new beam to your eye would be 
\[ l(\lambda) = i(\lambda)r_1(\lambda)r_2(\lambda) \]
we cannot predict this color from the observed colors of the individual paints!
for an example if \( r_2 \) is narrow band, it might be completely killing off all of \( r_1 \), or doing almost nothing to it.

Mathematical Model

this basic model was established using some perceptual experiments and math.
no microscopes
we will re-derive this model, and get a clearer idea about color as a vector space.
to do experiments we use a basic setup (see fig)
ask user if two patches match
simple scene \( \Rightarrow \) perceived color is the same as retinal.

transitivity

In our very first experiment, we test that the metameric relation is transitive
In particular we find that, if \( l_1(\lambda) \) is indistinguishable to \( l'_1(\lambda) \), and \( l'_1(\lambda) \) is indistinguishable to \( l''_1(\lambda) \), then \( l'_1(\lambda) \) will always be indistinguishable to \( l''_1(\lambda) \).
Due to this transitivity, we actually define \( \bar{c}(l_1(\lambda)) \), “the color of the beam \( l_1(\lambda) \)”, as the collection of light beams that are indistinguishable to a human observer from \( l_1(\lambda) \).
in this language \( \bar{c}(l_1(\lambda)) = \bar{c}(l'_1(\lambda)) = \bar{c}(l''_1(\lambda)) \).
a (retinal) color is an equivalence class of light beams

linear structure

we want to use linear algebra to work on colors
this will give us a theory of primary light mixing
- different from paint mixing.
but this will take a little bit of abstraction.
We know from physics that when two light beams, \( l_1(\lambda) \) and \( l_2(\lambda) \), are added together, they simply form a combined beam with light distribution \( l_1(\lambda) + l_2(\lambda) \).
Thus, we attempt to define the addition of two colors, as the color of the addition of two beams.
\[ \bar{c}(l_1(\lambda)) + \bar{c}(l_2(\lambda)) := \bar{c}(l_1(\lambda) + l_2(\lambda)) \]
For this to be well defined, we must experimentally verify that it does not make a difference which beam we choose as representative for each color.
- if \( \bar{c}(l_1(\lambda)) = \bar{c}(l'_1(\lambda)) \), then we must verify that, for all \( l_2(\lambda) \), we have \( \bar{c}(l_1(\lambda) + l_2(\lambda)) = \bar{c}(l'_1(\lambda) + l_2(\lambda)) \)
experiment confirms!

nonneg scalar mult

since we can multiply a light beam by a nonneg scalar, we try the definition
\[ \alpha \bar{c}(l_1(\lambda)) := \bar{c}(\alpha l_1(\lambda)) \]
Again, we need to verify that the behavior of this operation does not depend on our choice of beam.
• experiment confirms

annoying technicality
• we do not have a definition for scalar multiply with a negative number.
  – since there is no such thing as negative light
  – later we will see that we will really need negative combinations to create a desired color.
• let’s think about subtraction: when we say \( \vec{c}_1 - \vec{c}_2 = \vec{c}_3 \), we could define this to mean \( \vec{c}_1 = \vec{c}_3 + \vec{c}_2 \).
  – negatives jump over the equal sign and give us something well defined.
• now we can give meaning to negative colors
• actual and negative colors will be part of a larger space of things called extended colors, which will be a full linear space.

extended color space
• let us call any of our original equivalence classes of light beams using the term: actual color.
• Let us define an extended color as a formal expression of the form
  \[ \vec{c}_1 - \vec{c}_2 \]
  where the \( \vec{c} \) are actual colors.
• We define two extended colors \( \vec{c}_1 - \vec{c}_2 \) and \( \vec{c}_3 - \vec{c}_4 \), to be equal if \( \vec{c}_1 + \vec{c}_4 = \vec{c}_3 + \vec{c}_2 \).
• Any extended color that is not an actual color will be called an imaginary color.

ecs ops
• addition is \((\vec{c}_1 - \vec{c}_2) + (\vec{c}_3 - \vec{c}_4) := (\vec{c}_1 + \vec{c}_3) - (\vec{c}_2 + \vec{c}_4)\).
• Multiplication by \(-1\) is \(- (\vec{c}_1 - \vec{c}_2) := (\vec{c}_2 - \vec{c}_1)\)
• With these operations, we indeed have a linear space of extended colors!
• some extended colors are actual colors.
• we will typically drop the word “extended’
• going back to figure, vectors inside the cone are actual colors, while vectors outside the cone are imaginary colors.

Color Matching
• goal1: establish the dimensionality of color space
• goal2: give us a form for mapping light beams to color coordinates
• user watches two screens
• on left they are shown a a pure test beam \( l_\lambda \)
• on the right, they observe a light that is made up of positive combinations of three pure matching beams, with wavelengths 435, 545 and 625 nanometers.
• observer must adjust three intensity knobs on the right side, \( k_{435}(\lambda) \), \( k_{545}(\lambda) \) and \( k_{625}(\lambda) \) to get a match
• If the user cannot succeed, then they are allowed to move one or more of the matching beams over to the left side
  – like letting the intensity become negative.
• This process is repeated for all \( \lambda \)
• webdemo

results
• user can indeed succeed in obtaining a match for all visible wavelengths.
• so color space is 3D
• we get 3 so-called matching functions $k_{435}(\lambda)$, $k_{545}(\lambda)$ and $k_{625}(\lambda)$, (see Figure)
• Notice that, at each of the wavelengths 435, 545, and 625, one of the matching functions is set to 1, while the other two are set to 0.
• in summary

$$\vec{c}(l_{\lambda}) = [\vec{c}(l_{435}) \vec{c}(l_{545}) \vec{c}(l_{645})]$$

$$\begin{bmatrix}
  k_{435}(\lambda) \\
  k_{545}(\lambda) \\
  k_{625}(\lambda)
\end{bmatrix}$$

• and for mixed beams we get

$$\vec{c}(l(\lambda)) = [\vec{c}(l_{435}) \vec{c}(l_{545}) \vec{c}(l_{645})]$$

$$\begin{bmatrix}
  \int_{\Omega} d\lambda \ l(\lambda) \ k_{435}(\lambda) \\
  \int_{\Omega} d\lambda \ l(\lambda) \ k_{545}(\lambda) \\
  \int_{\Omega} d\lambda \ l(\lambda) \ k_{625}(\lambda)
\end{bmatrix}$$

• and we can compute the mapping from light to color

visualize

• we can visualized like LMS space
• lasso passes through axes
• lasso does leave the first octant

summary

• light comes in beams $l(\lambda)$.
• actual colors are equivalence classes of metameric beams
• these live inside of the mathematical “color space”
  - color space includes imaginary colors as well.
• this is a 3d linear space
• one basis is $[\vec{c}(l_{435}) \vec{c}(l_{545}) \vec{c}(l_{645})]$
• we have an equation to compute color coordinates in this basis given a beam.
• This linear space is consistant with beam addition
• note: this linear space is not consistant with paint mixing!

Bases

• we can insert any (non singular) 3-by-3 matrix $M$ and its inverse to obtain

$$\vec{c}(l(\lambda)) = ([\vec{c}(l_{435}) \vec{c}(l_{545}) \vec{c}(l_{645})]M^{-1}) \begin{bmatrix}
  \int_{\Omega} d\lambda \ l(\lambda) \ k_{435}(\lambda) \\
  \int_{\Omega} d\lambda \ l(\lambda) \ k_{545}(\lambda) \\
  \int_{\Omega} d\lambda \ l(\lambda) \ k_{625}(\lambda)
\end{bmatrix}$$

$$= [\vec{c}_1 \ \vec{c}_2 \ \vec{c}_3] \begin{bmatrix}
  \int_{\Omega} d\lambda \ l(\lambda) \ k_1(\lambda) \\
  \int_{\Omega} d\lambda \ l(\lambda) \ k_2(\lambda) \\
  \int_{\Omega} d\lambda \ l(\lambda) \ k_3(\lambda)
\end{bmatrix}$$

• where the $\vec{c}_i$ describe a new color basis defined as

$$[\vec{c}_1 \ \vec{c}_2 \ \vec{c}_3] = [\vec{c}(l_{435}) \vec{c}(l_{545}) \vec{c}(l_{645})]M^{-1}$$
the \( k(\lambda) \) functions form the new associated matching functions, defined by
\[
\begin{bmatrix}
  k_1(\lambda) \\
  k_2(\lambda) \\
  k_3(\lambda)
\end{bmatrix}
= M
\begin{bmatrix}
  k_{435}(\lambda) \\
  k_{545}(\lambda) \\
  k_{625}(\lambda)
\end{bmatrix}
\]

**basis specification**

- Starting from any fixed basis for color space, such as \( \vec{c}(l_{435}) \vec{c}(l_{545}) \vec{c}(l_{645}) \),
- 1: specify an invertible 3-by-3 matrix \( M \).
- 2: somehow specify three actual colors \( \vec{c}_i \).
  - Each such \( \vec{c}_i \) can be specified by some light beam \( l_i(\lambda) \) that generates it.
  - plug each such light beam into above calculation to obtain its 456 color coordinates, determining the matrix.
  - plugging in each new light gives you one column of \( M^{-1} \).
- directly specify three new matching functions.
  - To be valid matching functions, they must arise from a basis change like the above Equation, and so each matching function must be some linear combination of \( k_{435}(\lambda) \), \( k_{545}(\lambda) \) and \( k_{625}(\lambda) \)
  - each new matching function corresponds to a row of \( M \).
  - else we will not respect metamerism
  - some cameras can mess this up
  - see fig

**LMS revisited**

- the LMS process we saw earlier looks just like this experiment-derived process.
- in fact, using the experiment derived process, it was conjectured that there must be 3 types of retinal cells that behave like the LMS process.
- this was only confirmed much later!
- in this context, the LMS matching functions we saw earlier give us describe what we are now calling a basis for color space.
- the coordinates of a color are called \( [L,M,S]^t \).
- The actual basis is made up of three colors we can call \( [\vec{c}_l, \vec{c}_m, \vec{c}_s] \).
- The color \( \vec{c}_m \) is a very imaginary color
  - there is no real light beam with LMS color coordinates \( [0, 1, 0]^t \).

**Gamut**

- observe: we cannot find three vectors that are contained in the horseshoe cone, while also containing the entire cone in their positive span.
- so if we want a basis where all actual colors have non-negative coordinates, at least one of the basis vectors must lie outside of the cone of actual colors.
  - Such a basis vector must be an imaginary color.
- Conversely, if all of our basis vectors are actual colors, and thus within the color cone, then there must be some actual colors that cannot be written with non-negative coordinates
- in this basis. We say that such colors lie outside the gamut of this color space.

**typical color space descriptions**

- one finds 3 actual “primary” lights, which have actual colors, and uses that as a basis.
– this will have a limited gamut, and some actual colors will have negative coordinates.
– if we want to display colors with our primaries, we may have to clamp out the negatives.

• alt, one describes three (appropriate) matching functions. in this case, it might be natural to have these functions be non-negative.
  – like if i am building a camera.
  – in this case, all actual colors will have positive coordinates.
  – mathematically, this means that at least one of the associated basis colors must be imaginary.

**XYZ space**

• central standardized space
• specified by the three matching functions called \( k_x(\lambda) \), \( k_y(\lambda) \) and \( k_z(\lambda) \), (see figure).
• The coordinates for some color with respect to this basis is given by a coordinate vector that we call \([X, Y, Z]^t\).
• These particular matching functions were chosen such that they are always positive, and so that the \( Y \)-coordinate of a color represents its overall perceived “luminance”. Thus, \( Y \) is often used as a black and white representation of the color.
• The associated basis \([\vec{c}_x, \vec{c}_y, \vec{c}_z]\) is made up of three imaginary colors; the axes in are outside of the color cone.

**RGB**

• there are a variety of RGB standards
• current one is called Rec. 709 RGB space.
• basis \([\vec{c}_r, \vec{c}_g, \vec{c}_b]\) is made up of three actual colors intended to match the colors of the three phosphors of an ideal monitor/tv display.
• Colors with non-negative RGB coordinates can be produced on a monitor and are said to lie inside the gamut of the color space. These colors are in the first octant of the Figure.
• some actual colors lie outside the gamut
• Additionally, on a monitor, each phosphor maxes out at “1”, which also limits the achievable outputs.
• images with colors outside the gamut need some kind of mapping/clipping to keep in the gamut. (advanced topic)
• later we see another type of animal called sRGB