1 Example

Exercise 1. Suppose $T(1) = 3$ and $T(n) = 3T(n/2) + n$. How would you find $T(8)$? The point of this exercise is the process.

Solution
Expand and substitute using the formula for the recurrence:

\[
T(8) = 3T(4) + 8 = 3 [3T(2) + 4] + 8 = 9T(2) + 20 = 9 [3T(1) + 2] + 20 = 27T(1) + 38 = 119
\]

This is the same approach that’s used to prove the Master Theorem.

2 Master Theorem

Start with a recurrence $T(n) = aT(n/b) + cn^k$ (supposing that $T(p_0) = q_0$ for constants $p_0$ and $q_0$) and expand:

\[
T(n) = aT(n/b) + cn^k = a \left( aT(n/b^2) + c \left( \frac{n}{b^2} \right)^k \right) + cn^k = a^2T(n/b^2) + cn^k \left( 1 + \frac{a}{b^k} \right)
\]

\[
\vdots
\]

\[
= a^sT(n/b^s) + cn^k \left[ \left( \frac{a}{b^k} \right)^s + \left( \frac{a}{b^k} \right)^{s-1} + \ldots + \frac{a}{b^k} + 1 \right]
\]

We stop expanding when we reach the base case, when $\frac{n}{b^s} = p_0$. This occurs after $s \approx \log_b \left( \frac{n}{p_0} \right) = \log_b n + \text{constant iterations}$. Notice that the expression is split into two terms. The asymptotic form of $T(n)$ is just a competition between these two terms to see which one dominates.

The second term has a geometric sum: using the formula for a geometric sum gives:

\[
T(n) = a^s q_0 + cn^k \left[ \frac{1 - \left( \frac{a}{b^k} \right)^{s+1}}{1 - \frac{a}{b^k}} \right]
\]

Exercise 2. Use the above expansion to derive the case of the Master Theorem for $a < b^k$.

Solution
Here, $\frac{a}{b^k} < 1$, and as $n$ (and therefore $s$) grows large the sum of the above geometric series is
dominated by the constant term \( \frac{1}{\log_b a} = \Theta(1) \). So \( T(n) = \Theta(n^s) + \Theta(n^k) \). Using our expression for \( s \):

\[
a^s = \Theta(a^{\log_b n}) = \Theta(n^{\log_b a}) = o(n^k)
\]

since \( a < b^k \) means that \( \log_b a < k \). We therefore get that \( T(n) = o(n^k) + \Theta(n^k) = \Theta(n^k) \).

**Exercise 3.** Now derive the Master Theorem for \( a > b^k \).

**Solution**
Proceeding like the previous case, the geometric sum is now dominated by the:

\[
\frac{\left(\frac{a}{b^k}\right)^{s+1}}{\frac{a}{b^k} - 1} = \Theta \left( \left( \frac{a}{b^k} \right)^s \right)
\]

term. Then the second term of \( T(n) \) is:

\[
 cn^k \cdot \Theta \left( \left( \frac{a}{b^k} \right)^{\log_b n} \right) = cn^k \cdot \Theta \left( \frac{n^{\log_b a}}{n^k} \right) = \Theta \left( n^{\log_b a} \right)
\]

This along with the result from the previous exercise that \( a^s = \Theta(n^{\log_b a}) \) gives that \( T(n) = \Theta(n^{\log_b a}) \).

**Exercise 4.** Derive the Master Theorem for \( a = b^k \).

**Solution**
Every term in the geometric series is now 1. There are \( s + 1 \) terms, so the second term of \( T(n) \) becomes:

\[
 cn^k(s + 1) = \Theta \left( n^k \log_b n \right) = \Theta \left( n^k \log n \right)
\]

The first term of \( T(n) \) is \( \Theta(n^{\log_b a}) = \Theta(n^k) \) so the second term dominates and \( T(n) = \Theta(n^k \log n) \).

Qualitatively, if \( a > b^k \), the bottleneck of the recurrence is the number of recursive calls we have to make. Otherwise, it’s the extra work done during each call (i.e. the \( cn^k \) term) that dominates the runtime.