Problem 1

A vertex cover is a set of vertices such that every edge in the graph has at least one endpoint in the set. In a graph $G = (V, E)$, $C$ is a vertex cover iff $V - C$ is an independent set. Show that for every constant $c$, there exists a graph for which even if we have a set $C$ that’s a $2$-approximation of the Minimum Vertex Cover, the corresponding independent set $V - C$ is not within a factor of $c$ of the Maximum Independent Set.

Solution: Consider a cycle of $n$ vertices. The minimum vertex cover has size $\lceil \frac{n}{2} \rceil$ and the maximum independent set has size $\lfloor \frac{n}{2} \rfloor$. However, the approximation algorithm for Minimum Vertex Cover could give a vertex cover of size $n - 1$ since $n - 1 \leq 2\lceil \frac{n}{2} \rceil$. Then, the corresponding independent set would have size $1$, which would be off by a factor of $\lfloor \frac{n}{2} \rfloor$. This works for any $c$ by taking $n \to \infty$.

Problem 2

Consider the nearest-neighbor heuristic for the Euclidean Traveling Salesman Problem: Start from any vertex and repeatedly move to the closest unvisited vertex. Once all vertices have been visited, return to the initial vertex. Give an example where the nearest-neighbor heuristic returns a longer tour than optimal.

Solution: There are multiple examples that work; one such example is $A = (0,0), B = (0,1), C = (1,1), D = (1,2)$:
The nearest-neighbor heuristic starting from vertex $A$ gives $A \rightarrow B \rightarrow C \rightarrow D \rightarrow A$ with a length of $3 + \sqrt{5}$ while the optimal tour is $A \rightarrow B \rightarrow D \rightarrow C \rightarrow A$ with a length of $2 + 2\sqrt{2}$ (note that $\sqrt{5} > 2$ but $2\sqrt{2} < 3$).

**Problem 3:** Now, consider a similar heuristic for Euclidean TSP:

1. Start with a trivial cycle of one vertex.
2. At each step, find the vertex $v$ with the minimum distance to any vertex in the cycle. Let $u$ be the vertex already on the cycle closest to $v$.
3. Add $v$ to the cycle after $u$.

Show that this heuristic gives a 2-approximation for Euclidean TSP.

**Solution:** Suppose there are $n$ vertices in our graph. Let $H_i$ be the cycle formed at step $i$ by adding vertex $v_i$ to the cycle after $u_i$. Observe that $w(H_i) \leq w(H_{i-1}) + 2w(u_i, v_i)$. This is because adding $2w(u_i, v_i)$ is equivalent to going from $u_i$ to $v_i$ and then back to $u_i$, but once we’ve added $v_i$, our cycle skips directly to the next vertex instead of repeating $u_i$, which either preserves or reduces the length by the triangle inequality. Therefore the weight of our tour is $w(H_n) \leq 2\sum_{i=1}^{n} w(u_i, v_i)$.

However, note that we are adding vertices to our cycle in the same order as in Prim’s algorithm for finding the MST of a graph. Therefore, edges $(u_1, v_1), \ldots, (u_n, v_n)$ form an MST $T$ of our graph. Letting $H^*$ denote the optimal tour, we have

$$w(H_n) \leq 2w(T) \leq 2w(H^*)$$

which shows that we have a 2-approximation.