lecture 17

Topics:
- Where are we now?
- Impulse and elastic collisions
- Rigid bodies are weird
- The angular velocity vector
- An impulsive demo

Where are we now?

We began our discussion of rigid body rotations by discussing the simple case of rotations about a fixed axis. Today, I will spend a little time discussing in general the important ideas of the angular velocity vector and the reference point. I am also going to discuss a nice example of motion in a plane, which introduces the idea of impulse that we will use to explore more complicated situations.

Impulse and elastic collisions

The animation (RODBOUNC.EXE) on the screen shows a rigid rod in the $y$-$z$ plane in a gravitational field bouncing completely elastically on a frictionless surface. This problem is a nice example of the use of impulsive forces and torques to solve problems. I will start by analyzing this example to show you how the animation was produced.

We will assume that the rod is initially either not rotating at all (this is what is shown in the animation) or rotating in a vertical plane. If so, the motion stays in the same vertical plane, as long as the rod is perfectly symmetrical and the plane on which it bounces is perfectly flat. Thus we can analyze it without worrying about the full three dimensional complexity of angular momentum. Then we can just choose our coordinate system so that $\hat{z}$ is vertical and the rod is bouncing in the $y$-$z$ plane, as we assumed.

Except when the rod is actually in contact with the frictionless surface, the motion is extremely simple. The rod rotates with some fixed angular velocity $\omega_i$ — the subscript $i$ is for “initial” (the axis $\hat{x}$ is out of the plane in the $x$ direction) and the center of mass, in the center of the rod rises and falls in the constant gravitational field, so that when one end of the rod hits the frictionless surface, the center of mass is moving with some velocity $v_i$ in the vertical direction ($\vec{v} = v\,\hat{z}$), which will usually be negative, but not always, because the rotation of the rod may cause a collision even if the center of mass is rising. When a collision occurs, we get a new velocity and angular velocity, $v_f$ and $\omega_f$ (subscript $f$ for “final”). Our job is to calculate $v_f$ and $\omega_f$ in terms of $v_i$ and $\omega_i$ — then we can follow the system until the next collision and do it again, and so on until we get tired. Or better still, we can simply program it into the animation and watch the pretty bouncing rod until we get mesmerized.

Suppose that the rod has mass $m$ and length $2\ell$. Suppose further that the collision occurs with
the rod at an angle $\theta$ (between 0 and $\pi$) from the horizontal, as shown below.

$$\int \vec{F} \, dt = \hat{z} \int \vec{F} \, dt$$

(1)

Also shown is the translational velocity of the center of mass and the rotational motion, $\omega \hat{z} \times \vec{r}$, which gets added on to the motion of the center of mass to produce the full motion of the end of the rod.

During the collision, there is a force, $\vec{F}$, on the rod from the frictionless surface. Because the surface is frictionless, the force is purely vertical, $\vec{F} = F \hat{z}$. Now because the bounce happens very quickly, we can ignore the motion of the rod while the bounce is taking place. Then all that matters is the integral of the force over the period of the bounce,

$$\int_{\text{bounce}} dt \vec{F} = \hat{z} \int_{\text{bounce}} dt F$$

(2)

This is called the “impulse.” Now the point is that the impulse does double duty. 1 — It changes the linear momentum of the center of mass. Because the force is the rate of change of the momentum, the impulse is the total change in the momentum:

$$m (v_f - v_i) = \int_{\text{bounce}} dt \frac{dp}{dt} = \int_{\text{bounce}} dt F$$

(3)

2 — It also changes the angular momentum about the center of mass. Because the torque is the rate of change of angular momentum, the cross product of the lever arm with the impulse is the total change in the angular momentum. The rod has length $2\ell$, so this looks like

$$I (\omega_f - \omega_i) = \int_{\text{bounce}} dt (\vec{r} \times \vec{F})_x = \left( \vec{r} \times \int_{\text{bounce}} dt \vec{F} \right)_x = -\ell \cos \theta \int_{\text{bounce}} dt F$$

(4)

Let me emphasize again the key step here. Because we have assume that the bounce takes place very quickly, we can ignore the motion of the rod while the bounce is taking place. That allows us to take the $\vec{r}$ out of the integral in (4). Then the change in $v$ and the change in $\omega$ are related —

$$\ell \cos \theta m (v_f - v_i) + I (\omega_f - \omega_i) = 0$$

(5)

In addition to (5), we know that energy is conserved. The energy is the kinetic energy in motion of the center of mass and rotation. Thus conservation of energy is

$$\frac{1}{2} m v_i^2 + \frac{1}{2} I \omega_i^2 = \frac{1}{2} m v_f^2 + \frac{1}{2} I \omega_f^2$$

(6)
At this point, we could plug this into Maple or Mathematica and ask the computer to solve for \( v_f \) and \( \omega_f \) in terms of \( v_i \) and \( \omega_i \). But there is a useful trick involved in doing it by hand, so let’s go on a while. First write energy conservation as

\[
m (v_f^2 - v_i^2) + I (\omega_f^2 - \omega_i^2) = 0
\]

(7)

Now we can factor this

\[
m (v_f - v_i) (v_f + v_i) + I (\omega_f - \omega_i) (\omega_f + \omega_i) = 0
\]

(8)

and now use (5) to write this as

\[
m (v_f - v_i) (v_f + v_i) - \ell \cos \theta m (v_f - v_i) (\omega_f + \omega_i) = 0
\]

(9)

The trick here is pretty general. We know there is a solution to the twin equations (5) and (6) of the form \( v_f = v_i \) and \( \omega_f = \omega_i \) because this satisfies (5) and if nothing changes, energy is conserved. We have just written (6) so that this solution is manifest. Of course, we are not interested in the case because this is not what happens in the collision. There must always be some force on the rod during the collision, so we always get a non-trivial change in \( v \) and \( \omega \). But writing (6) this way allows to eliminate the trivial solution. Thus for the physical solution we are interested in, we must have

\[
(v_f + v_i) - \ell \cos \theta (\omega_f + \omega_i) = 0
\]

(10)

This is now another linear equation for \( v_f \) and \( \omega_f \), so we can easily solve (5) and (10), and the result is

\[
v_f = \frac{(m \ell^2 \cos^2 \theta - I) v_i + 2 \ell \ell \cos \theta \omega_i}{m \ell^2 \cos^2 \theta + I} \quad \omega_f = \frac{2 m \ell \cos \theta v_i - (m \ell^2 \cos^2 \theta - I) \omega_i}{m \ell^2 \cos^2 \theta + I}
\]

(11)

It is this that we have used to construct the animation.

Nothing we have done depends on the precise value of \( I \). For a solid rod, \( I = m \ell^2 / 3 \), but we don’t have to look only at that case. It is interesting to look at this for various \( I \)s. One very interesting limit is \( I = m \ell^2 \), which corresponds to a dumbbell, with the masses at the ends of a light rod. This actually allows us to use our physical intuition to get a nice check of (11). Suppose that for a dumbbell, \( \cos \theta \) is close to zero when the left mass hits the surface. Because the force of the rod on the masses is nearly horizontal in this case, it has very little effect of the motion of the two masses. Thus we expect the left mass to bounce and simply reverse its velocity, and the right mass to just keep going. Now for \( \theta \approx 0 \), the motion of the masses is nearly in the vertical direction and the vertical components are approximately

\[
v_{\text{left}} \approx v - \ell \omega \quad v_{\text{right}} \approx v + \ell \omega
\]

(12)

Thus to get \( v_{\text{left}} \) to change sign while \( v_{\text{right}} \) to remain unchanged, we want the bounce to approximately interchange \( v \) and \( \ell \omega \),

\[
v_f \rightarrow \ell \omega_i \quad \ell \omega_f \rightarrow v_i
\]

(13)
It is easy to see from (11) that this is what happens.

In the case of the rigid rod, with smaller moment of inertia, the other end of the rod actually moves faster after such a nearly horizontal bound. Again a limit may make clearer what is going on. The limit $I \to 0$ corresponds to a mass in the center of the light rod. In this case, for $\theta \approx 0$, we expect the center of mass to keep moving, $v_f \approx v_i$, which again accords with (11).

Meanwhile, notice that when the rod is rotating a lot, it doesn’t go up as far — this is because more of the energy is stored in rotational kinetic energy and there is less in center of mass motion after the collision.

**Rigid bodies are weird**

So what’s the big deal about torque and angular momentum. Surely, this is just like force and momentum. You push something and it moves (or accelerates, at least). You twist something and it turns. But torque equals rate of change of angular momentum implies some pretty remarkable things. When you twist a spinning rigid body carrying a large angular momentum, the twist does very counter-intuitive things because the twist does not directly change the orientation of the body. Instead, what a torque does is to change the direction of the angular momentum. And the direction of the angular momentum is tied not to the orientation of the body, which is constantly changing, but to the orientation of the rotation axis. A gyroscope is the most familiar example of this. A torque that one would naively think would cause the body to fall instead causes it to precess. This is very familiar, but it is worth seeing over and over again. Here it is for a simple top. Take a bicycle wheel and weight the rim with lead. Get it spinning with speed $v$. The angular momentum is then approximately $mvr$ where $r$ is the radius of the wheel and $m$ is the total mass. This would be exactly right if all of the mass were concentrated in the rim at radius $r$.

\[ \vec{L} \quad \text{(14)} \]

Now if we apply a torque to the handle, which is the axis of rotation, strange things happen because we are actually changing the angular momentum. Precession is one example.

\[ \vec{F} \quad \text{(15)} \]
The torque in the diagram is into the paper. Thus the change in the angular momentum is into the paper. But the only way that can happen is if the direction of the angular momentum changes — and the orientation of the handle must go with the angular momentum — so the system precesses. You see that precession is easy to explain in terms of torque and angular momentum, but perhaps not so easy to understand in your bones.

Here is another situation which is basically the same, but which I find even stranger. If I stand on a turntable with the wheel axis horizontal, and I try to twist the handle so that the angular momentum of the wheel points slightly down, to conserve angular momentum, I will have to start spinning in the counterclockwise direction, and develop angular momentum upwards. This is pretty weird, because it means that by trying to produce a torque in one direction (horizontal) I have actually produced a torque in the vertical direction.

What is going on here??????

I find this sufficiently strange that I want to show you how it comes about in a particular very simple case.

Consider a light rigid frame of crossed bars with weights of equal mass $m$ on two of the opposite ends. The two masses form a dumbbell rotating in the $x$-$y$ plane. The cross piece is supported in two frictionless sleeves that allow the system to rotate, but can be used to supply a torque. This is shown below in the $x$-$z$ plane:

\[
\vec{L} = 2mv\ell \hat{z}
\]

(16)

If at some time $t = 0$, the upper mass at $x = \ell$ is moving in the $+y$ direction with speed $v$, and the lower one at $x = -\ell$ in the $-y$ direction with the same speed, then the angular momentum of the system is $2mv\ell \hat{z}$.

Now suppose that at time $t = 0$, we apply a large torque $N$ in the $+y$ direction for a very short time $\Delta t$, short enough that we can neglect the motion of the masses during the time. We actually supply this torque by twisting the frictionless sleeve.

\[
\vec{F} - \vec{F}
\]

(17)
but the effect is the same as applying the forces to the masses (because of the rigidity of the frame)

\[ F' = \frac{N}{2\pi} \]

The magnitude of the force is fixed by the value of the torque.

Now the torque in (18) changes the direction of the motion of the two masses. The change in the momentum is \( F' \Delta t \). From the top, in the \( y-z \) plane, the resulting momentum looks like this (with the momenta of the bottom mass dashed and the masses not shown)

\[ \text{(19)} \]

Now as the masses move in their new direction, they drag the rest of the rigid body along with them, establishing a new axis of rotation, shown as the dotted line below.

\[ \text{new rotation axis} \]

(20)

Thus the system must precess.

Of course, this all really happens at once, but it makes it easier for me to understand what is going on in precession if I disarticulate things and think first about a very quick application of torque changing the direction of the rotating masses, and then the rigid body forces that hold the system together pulling the rest of the system along with them. In the next couple of weeks, I am going to use this trick in more complicated ways to ease our way into the physics of rigid body rotations.
The angular velocity vector

We have seen that the velocity of point $\vec{r}_j$ on a rigid body rotating with angular velocity $\omega$ about an axis $\hat{n}$ through a point $\vec{r}_0$ is

$$\omega \, \hat{n} \times (\vec{r}_j - \vec{r}_0)$$  \hspace{1cm} (21)

The important point is that $\omega$ and $\hat{n}$ always appear in combination, as the product $\omega \, \hat{n}$. This is a vector with direction $\hat{n}$ and magnitude $\omega$. It is called the “angular velocity vector”

$$\vec{\omega} \equiv \omega \, \hat{n}$$  \hspace{1cm} (22)

Now here is a very important fact. Like any vector, $\vec{\omega}$ can be taken apart into components. Angular velocity vectors can be added and subtracted. This may not sound very remarkable, but in fact, ordinary rotations do not work this way. Unlike vector coordinates like the coordinates of the center of mass, the quantities that describe the orientation of a rigid body are not vectors. They are angles and they do not form a linear space. You can’t add them. The reason is simply that the structure of rotations is more complicated than the structure of translations. The order in which rotations are done matters to the final configuration. Because order doesn’t matter when you add numbers or vectors, that means that rotations cannot simply add. They must compose in some more complicated way.

Here’s a simple example. One good way to specify the orientation of a rigid body is by specifying a reference orientation and specifying the rotation required to get from the reference orientation to the actual orientation. A rotation, in turn, can be specified by giving the axis of rotation, and the magnitude of the rotation in radians. If you put together two rotations about the same axis, the magnitudes just add. But the trouble is that if the axes are different, the combination of the two rotations is a rotation about some new axis, by some magnitude that is a complicated function of the angles and axes. Both the axis and the magnitude of the combined rotation depend on which of the component rotations is performed first. This is what makes rotation of rigid bodies such a complicated and interesting subject.

For example, consider a regular tetrahedron with the four vertices labeled by different numbers, 1-4. Looking down on this tetrahedron, it might look like this:

$$\hat{a} \quad \hat{b} \quad \hat{c}$$  \hspace{1cm} (23)

where $\hat{a}$, $\hat{b}$ and $\hat{c}$ represent axes through the center of the tetrahedron (each of these vectors is coming slightly out of the plane of the paper).
If I do a rotation by $2\pi/3$ about the $\hat{a}$ axis, the tetrahedron rotates into itself with the numbers on the vertices rotated as $1 \rightarrow 2 \rightarrow 4 \rightarrow 1$, as shown:

$$
\begin{array}{c}
\hat{a} \\
\hat{b} \\
2 \\
\end{array} 
\rightarrow 
\begin{array}{c}
\hat{a} \\
\hat{b} \\
1 \\
\end{array} 
\rightarrow 
\begin{array}{c}
\hat{a} \\
\hat{b} \\
3 \\
\end{array} 
\rightarrow 
\begin{array}{c}
\hat{a} \\
\hat{b} \\
4 \\
\end{array} 
\rightarrow 
\begin{array}{c}
\hat{a} \\
\hat{b} \\
2 \\
\end{array} \quad (24)
$$

If I now do a rotation by $2\pi/3$ about the $\hat{b}$ axis, the tetrahedron rotates into itself with the numbers on the vertices rotated as $2 \rightarrow 3 \rightarrow 4 \rightarrow 2$, as shown:

$$
\begin{array}{c}
\hat{a} \\
\hat{b} \\
4 \\
\end{array} 
\rightarrow 
\begin{array}{c}
\hat{a} \\
\hat{b} \\
2 \\
\end{array} 
\rightarrow 
\begin{array}{c}
\hat{a} \\
\hat{b} \\
3 \\
\end{array} 
\rightarrow 
\begin{array}{c}
\hat{a} \\
\hat{b} \\
4 \\
\end{array} 
\rightarrow 
\begin{array}{c}
\hat{a} \\
\hat{b} \\
2 \\
\end{array} \quad (25)
$$

The result of these two rotations is equivalent to a single rotation by $4\pi/3$ (or $-2\pi/3$) around the $\hat{c}$ axis, as shown below

$$
\begin{array}{c}
\hat{a} \\
\hat{b} \\
2 \\
\end{array} 
\rightarrow 
\begin{array}{c}
\hat{a} \\
\hat{b} \\
1 \\
\end{array} 
\rightarrow 
\begin{array}{c}
\hat{a} \\
\hat{b} \\
3 \\
\end{array} 
\rightarrow 
\begin{array}{c}
\hat{a} \\
\hat{b} \\
4 \\
\end{array} 
\rightarrow 
\begin{array}{c}
\hat{a} \\
\hat{b} \\
2 \\
\end{array} \quad (26)
$$

On the other hand, if I do the same two rotations in the opposite order, something different happens. If I first do the rotation by $2\pi/3$ about the $\hat{b}$ axis, the tetrahedron rotates into itself with the numbers on the vertices rotated as $1 \rightarrow 3 \rightarrow 2 \rightarrow 1$, as shown:

$$
\begin{array}{c}
\hat{a} \\
\hat{b} \\
2 \\
\end{array} 
\rightarrow 
\begin{array}{c}
\hat{a} \\
\hat{b} \\
1 \\
\end{array} 
\rightarrow 
\begin{array}{c}
\hat{a} \\
\hat{b} \\
3 \\
\end{array} 
\rightarrow 
\begin{array}{c}
\hat{a} \\
\hat{b} \\
4 \\
\end{array} 
\rightarrow 
\begin{array}{c}
\hat{a} \\
\hat{b} \\
2 \\
\end{array} \quad (27)
$$
If I now do the rotation by $2\pi/3$ about the $\hat{a}$ axis, the tetrahedron rotates into itself with the numbers on the vertices rotated as $1 \rightarrow 4 \rightarrow 3 \rightarrow 1$, as shown:

The result of these two rotations is equivalent to a single rotation by $4\pi/3$ (or $-2\pi/3$) around an axis pointed down into the plane, as shown below.

So you see that for finite rotations, the order of the rotations makes a difference. There is no way that you can simply add the coordinates of vectors to get these results. Finite rotations are not vectors!

The reason that angular velocities are simpler is that they really only refer to infinitesimal rotations — $\omega$ is $d\theta/dt$. And infinitesimal rotations can be added without causing confusion. Technically, the reason that infinitesimal rotations are can be added like ordinary vectors is that the dependence on the order of two infinitesimal rotations is proportional to the product of the two infinitesimal angles, and can thus be ignored. Thus $d\bar{\omega}$ is a vector, even though a finite rotation is not. The relation between the angular velocity vector and the motion of the parts of a rigid body,

$$\bar{\omega} \times (\vec{r}_j - \vec{r}_0)$$

will be the crucial simple fact to hang onto as we explore the complicated world of rigid body rotations. This depends on not only the direction of the axis, but exactly where the axis is, which can be specified by specifying any point $\vec{r}_0$ on the axis. Any other point on the axis gives the same result, because, any other point has the form

$$\vec{r}_a = \vec{r}_0 + a\hat{n}$$

The extra term proportional to $\hat{n}$ doesn’t affect (30) because $\bar{\omega}$ and $\hat{n}$ are in the same direction so their cross product vanishes. You should remember that if you are going to add angular velocities, they must be defined with respect to the same reference point $\vec{r}_0$. Otherwise, things get messy.
An impulsive demo

An impulsive force is one that is applied for a very short time. To get a large change in momentum in a very short time requires a very large impulsive force. Applying a very large force is a good way of breaking things. Here is an example.