Why does a constant force on an object in a viscous medium produce a constant (terminal) velocity?

You know that the viscous drag force $F_{\text{drag}}$ is proportional to the velocity $v$ (and in the opposite direction); we’ll use the symbol $f$ to represent the coefficient of proportionality:

$$F_{\text{drag}} = -fv$$

When an object reaches terminal velocity, the drag force is equal and opposite to the constant external force, so the velocity is constant:

$$F_{\text{ext}} = -F_{\text{drag}} \quad \text{so} \quad v = \frac{F_{\text{ext}}}{f}$$

We seek a microscopic model for viscosity that can explain this observation. As the object is being pulled through the fluid by a constant force, the object is also subjected to numerous microscopic collisions with the molecules of the fluid. Let’s assume that these collisions are periodic: there is one collision per time $\Delta t$. Let’s also assume that as a result of each collision, our object’s velocity is “reset” to some random value $v_0$. Then, before the next collision, the object moves freely, influenced only by the constant external force. The constant force produces a constant acceleration $a = F_{\text{ext}}/m$, so the object’s velocity is given by the usual kinematic equation for constant acceleration:

$$v = v_0 + \left(\frac{F_{\text{ext}}}{m}\right)\Delta t$$

For a constant “downward” force (for instance, a gravitational force) the velocity of such a particle might look like this:
Take a close look at this graph. Each black circle shows an initial (random) velocity that follows a collision. On average, those velocities are zero (i.e. the random velocity is equally likely to be positive or negative). However, during the period between collisions, the object is subjected to a constant downward acceleration, so its velocity decreases linearly between each collision in a constant, predictable way. Thus, the average of all the velocities on the graph is a negative number: on average, the object is falling “down” with a constant velocity.

If we integrate the velocity we get the position as a function of time. The kinematic expression is:

\[ x = x_0 + v_0 t + \frac{1}{2} \left( \frac{F_{\text{ext}}}{m} \right) t^2 \]

and a graph of position versus time (using the velocities shown on the previous graph) is:

Each dot on the above graph shows one collision. Can you see that the object is in parabolic free-fall between each collision? The dashed line represents the average downward trajectory of the object. We see that, although the instantaneous motion of the object is a constant-acceleration trajectory (as expected for a constant force), the average motion is a constant-velocity trajectory (as observed in a viscous medium).

We can think of an object in motion in a viscous medium as experiencing “interrupted acceleration.” The object accelerates for a short time \( \Delta t \) . . . then it undergoes a collision, and its velocity is “reset” to some random value. Then it accelerates again for a short time . . . and its acceleration is interrupted again. The overall result of interrupted acceleration is motion at a constant velocity!
The overall average drift velocity can be obtained from the kinematic expression for position as a function of time, if we note that the average of \( v_o \) is zero:

\[
\langle v \rangle = \frac{x - x}{\Delta t} = \frac{1}{2} \left( \frac{F_{\text{ext}}}{m} \right) \Delta t
\]

Then, since the external force is related to the drift velocity by \( F_{\text{ext}} = f \langle v \rangle \), we have:

\[
\langle v \rangle = \frac{1}{2} \left( \frac{f \langle v \rangle}{m} \right) \Delta t
\]

in which we can cancel the drift velocities and rearrange to get an incredible relationship between the viscous drag coefficient \( f \) and the time between collisions \( \Delta t \):

\[
f = \frac{2m}{\Delta t}
\]

According to this equation, the macroscopic viscous drag coefficient of an object with mass \( m \) depends only on the frequency of microscopic collisions with that object! This relationship should make sense: if the collisions are very frequent (so \( \Delta t \) is small), then the viscous drag coefficient will be larger. Next time you pull a spoon out of a jar of honey, think about how the extremely rapid collisions between the “honey molecules” and the spoon give rise to the viscous drag you feel on the spoon!

We can immediately see the connection to Brownian motion if we remove the constant external force from our above discussion. Then the velocity as a function of time would look like this: the object moves at a constant random velocity over each time interval:

![Diagram](image-url)
Now, if we integrate the velocities to find the position, we see that the average position of the object is zero, but the object still “wanders” up and down over time:

Compare this graph of position with the previous one: in this case, the object moves with a constant velocity between each collision. This is an example of what physicists and mathematicians refer to as a random walk: the object moves randomly, taking a series of small “steps.” Each step can be in either direction (up or down, in this example).

Because a random walk is, by definition, random, we can only inquire about the average behavior of an object undergoing a random walk. For instance, we note that the average displacement of the object is always zero, since each step is equally likely to be up or down. Thus, in some sense, the object, on average, doesn’t “go anywhere.”

This observation is misleading, however, because the average distance from the origin will not be zero. We can better characterize the average distance by considering the mean square displacement, or \( <x^2> \). (We use this definition because we want the “distance” to always be positive, so we square the displacement to obtain a positive measure of distance.)

Why will the mean square displacement not be zero? Consider a random walk of four steps, where the steps are \( \Delta x_1, \Delta x_2, \Delta x_3, \text{ and } \Delta x_4 \). If the object starts at \( x = 0 \), then the final displacement of the object is given by \( (\Delta x_1 + \Delta x_2 + \Delta x_3 + \Delta x_4) \). What happens if we square that? We get a hideous mess that looks something like this:

\[
(\Delta x_1 + \Delta x_2 + \Delta x_3 + \Delta x_4)^2 = (\Delta x_1)^2 + (\Delta x_2)^2 + (\Delta x_3)^2 + (\Delta x_4)^2 + (\Delta x_1)(\Delta x_2) + (\Delta x_1)(\Delta x_3) + \ldots
\]
where we have omitted a number of the “cross terms” \((\Delta x_i)(\Delta x_j)\) in which \(i \neq j\). Now what can we say about the sign of all of these terms? We know that the squared terms such as \((\Delta x_1)^2\) will always be positive, regardless of the sign of \(\Delta x_1\). However, the cross terms will, on average, be equal to zero, because each cross term could have either a positive or negative sign, depending on the signs of the individual \(\Delta x_i\). Thus, the mean square displacement is given by dropping all of the cross terms in the above expression:

\[
\langle x^2 \rangle = \langle (\Delta x_1 + \Delta x_2 + \Delta x_3 + \Delta x_4)^2 \rangle = \langle (\Delta x_1)^2 \rangle + \langle (\Delta x_2)^2 \rangle + \langle (\Delta x_3)^2 \rangle + \langle (\Delta x_4)^2 \rangle
\]

The bottom line is that random displacements add in quadrature: the square of the overall displacement is equal to the sum of the squares of the individual displacements. You may recall a similar formula for adding standard deviations of random variables: the square of the standard deviation of the sum of a number of random variables is given by the sum of the squares of the individual standard deviations:

\[
\sigma^2 = (\sigma_1)^2 + (\sigma_2)^2 + (\sigma_3)^2
\]

We can simplify our random-walk model of Brownian motion by assuming that the object always takes a fixed step \(\Delta x\) at each time \(\Delta t\), and that only the direction of the step is random. In this case, if the object takes a total of \(N\) steps, the mean square displacement is:

\[
\langle x^2 \rangle = N(\Delta x)^2
\]

The total time required to take \(N\) steps is \(t = N\Delta t\), so we can rewrite this as:

\[
\frac{\langle x^2 \rangle}{t} = \frac{(\Delta x)^2}{\Delta t}
\]

Although this equation may look trivial (or even vacuous), its content is quite significant: the overall mean square displacement is directly related to the square of the step size \(\Delta x\). It is conventional to define a diffusion coefficient \(D\) by the relationship:

\[
D = \frac{(\Delta x)^2}{2\Delta t}
\]

where the factor of 2 is introduced for convenience. Thus, the mean square distance that an object will move as a result of Brownian motion during a time \(t\) is determined only by this diffusion coefficient:

\[
\langle x^2 \rangle = 2Dt
\]

Although we derived this expression by considering the microscopic motion of a particle, the diffusion coefficient is defined and measured by such macroscopic measurements.
In 1905, Einstein published a groundbreaking paper (no, not the one on special relativity or the one on the photoelectric effect, but another one!) in which he used the above arguments to provide support for the hypothesis that matter is made up of discrete atoms and molecules.

To complete the argument, we note that an object in thermal equilibrium at a temperature $T$ will have an average kinetic energy (in one dimension) equal to $\frac{1}{2}kT$, where $k$ is Boltzmann’s constant (equal to the ideal gas constant divided by Avogadro’s number, or $k = R/NA$), and $T$ is the temperature (in Kelvin):

$$\frac{1}{2}m\langle v_x^2 \rangle = \frac{1}{2}kT$$

Einstein pointed out that this relationship should hold for macroscopic objects as well as for atoms and molecules. We can solve for the mean square velocity to obtain:

$$\langle v^2 \rangle = \frac{kT}{m}$$

(we have dropped the subscript on $v$ for clarity; keep in mind that $v$ is only the component of the velocity in one direction). What is the mean square velocity of the Brownian object? If we recall that, for each step, the object travels a distance $\Delta x$ in a time $\Delta t$, then we can consider the mean square velocity to be the square of this ratio:

$$\langle v^2 \rangle = \frac{(\Delta x)^2}{(\Delta t)^2}$$

Since we defined the diffusion coefficient $D$ earlier as $D = (\Delta x)^2 / 2\Delta t$, we can rewrite this, and equate it to the expression for mean square velocity above:

$$\langle v^2 \rangle = \frac{D}{2\Delta t} = \frac{kT}{m}$$

to yield an expression for the time between collisions (the time of a single step, $\Delta t$):

$$\Delta t = \frac{2Dm}{kT}$$

Finally, we retrieve our earlier relationship between the coefficient of viscous drag ($f$) and the time between collisions, and insert our new expression for $\Delta t$ into that expression:

$$f = \frac{2m}{\Delta t} = \frac{2mkT}{2Dm} = \frac{kT}{D}$$

This expression, known as the **Einstein-Smoluchowski equation** can be re-written as:

$$Df = kT$$
What is remarkable about this equation is that it relates experimentally-measured, *macroscopic* properties—the diffusion coefficient $D$ and the coefficient of viscous drag $f$—with the *microscopic* thermal energy of a *single molecule*, which is $kT$.

Einstein pointed out that this theory depends on the assumption that the viscous liquid is made up of discrete molecules in constant thermal motion: the Brownian motion of the object results from the *discrete nature* of the surrounding medium. His theory made a specific *quantitative* prediction: if you measure $D$, $f$, and $T$ for a given system, you can calculate the value of Boltzmann’s constant $k$. Then, since $k = R/N_A$, and the value of the ideal gas constant is known (8.31 J/mol K), you can calculate Avogadro’s number:

$$Df = \left( \frac{R}{N_A} \right)T$$

so

$$N_A = \frac{RT}{Df}$$

In 1905, there were several numerical estimates of Avogadro’s number. Einstein noted that a measurement of Brownian motion should yield an *independent* determination of Avogadro’s number. If this new measurement of Avogadro’s number came close to the previous estimates, that observation would provide strong support for the hypothesis that “continuous fluids” are in fact made up of discrete molecules.

In your experiment, you will measure $<x^2>$ as a function of time for the Brownian motion of small, spherical, 1-µm colloid particles in water. Because these objects are spheres, the coefficient of viscous drag is given by Stokes’ Law: $f = 6\pi\mu r$. Thus, by measuring the motion of small spheres in water, you can obtain an estimate of Avogadro’s number! This experiment was seen by many physicists in the early 20th century as providing conclusive evidence for the existence of atoms and molecules.